Replacement Lemma and its proof

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Abstract

This note proves the Replacement Lemma that we talked about in class on Tuesday 2/19/2002.

Lemma 0.1 (Replacement) *If*

1. $D \triangleright \Gamma \vdash E[e] : t$*, such that the hole in* E *occurs at position* p
2. $D' \triangleright \Gamma \vdash e : t'$
3. D' *is a subderivation of* D *occurring at position* p *and*
4. $\Gamma \vdash e' : t'$

then, $\Gamma \vdash E[e'] : t$.

It is crucial that D' be a subderivation of D. For otherwise, the hypotheses

1. $\Gamma \vdash E[e] : t$,
2. $\Gamma \vdash e : t'$ and
3. $\Gamma \vdash e' : t'$

do not imply $\Gamma \vdash E[e'] : t$. Consider the counter-example $E = \Box$, $e =$ *DivZero*, $e' = 5$, $t =$ `bool` and $t' =$ `int`. The judgements

1. $\Gamma \vdash \Box[\underline{\textit{DivZero}}] : \texttt{bool}$,
2. $\Gamma \vdash \underline{\textit{DivZero}} : \texttt{int}$ and
3. $\Gamma \vdash 5 : \texttt{int}$

are all true but imply $\Gamma \vdash \Box[5] : \texttt{int}$, which is false.

Also, it is necessary that the position of D' in D and the position of e in E be the same. Otherwise, we have the counter-example $E[\underline{\textit{DivZero}}]$, where $E =$ **if** $\Box$ **then** $\underline{\textit{DivZero}}$ **else** 1. If

1. $D \rhd \emptyset \vdash e : \texttt{int}$
2. $D_1 \rhd \emptyset \vdash \underline{\textit{DivZero}} : \texttt{bool}$ and
3. $D_2 \rhd \emptyset \vdash \underline{\textit{DivZero}} : \texttt{int}$

then both D_1 and D_2 are subderivations of D. If the restriction about the subderivation D' being at position p were removed, then choosing D' to be D_2 means that the propositions

1. $D \rhd \emptyset \vdash E[\underline{\textit{DivZero}}] : \texttt{int}$
2. $D_2 \rhd \emptyset \vdash \underline{\textit{DivZero}} : \texttt{int}$
3. D_2 is a subderivation of D and
4. $\emptyset \vdash 5 : \texttt{int}$

are all true, but imply the false judgement $\emptyset \vdash E[5] : \texttt{int}$.

Proof (of **Replacement Lemma**)

By induction on D. For the base cases, D has exactly one node. Therefore, $E = \Box$, $D' = D$, and $t = t'$ and the result follows.

For the inductive cases, we have the following subcases depending on E:

1. $E = \Box$. This implies $D = D'$ and $t = t'$ and this is similar to the case above.

2. $E = +(E_1, e_2)$: By the Inversion Lemma, $t = \mathtt{int}$, and there are derivations D_1 and D_2 such that

$$D = \text{ AOP } \frac{D_1 \rhd \Gamma \vdash E_1[e] : \mathtt{int} \qquad D_2 \rhd \Gamma \vdash e_2 : \mathtt{int}}{\Gamma \vdash +(E_1[e], e) : \mathtt{int}}$$

   It follows that D' is a subderivation of D_1. Clearly, D_1 is a proper subderivation of D. Thus, by the induction hypothesis, $\Gamma \vdash E_1[e'] : \mathtt{int}$. Again, by the Inversion Lemma, $\Gamma \vdash e_2 : \mathtt{int}$. The result follows from the application of the AOP rule.

The other cases for E are similar and omitted. $\dashv$