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Gravitational Instantons[1]

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Abstract

The path-integral approach to quantum field theory assigns special importance to finite action Euclidean solutions of classical field equations. In Yang–Mills gauge theories, the instanton solutions of classical field equations with self-dual field strength have given rise to a new, nonperturbative treatment of the quantum field theory and its vacuum state. Since gravitation is also a species of gauge theory, one might think that similar phenomena would occur in gravity. The authors recently sought and found a new self-dual solution to Euclidean gravity which plays a role parallel to that of the Yang–Mills instanton. Gravitational instantons now promise to yield new insights into the nature of quantum gravity.

§(1): *Introduction*

Einstein's theory of gravitation is one of the most beautiful structures of classical relativistic physics. Being a classical field theory, however, gravity has proved to be very difficult to be incorporated in the context of modern quantum field theory. The desire to treat gravitation as a quantum field resembling other elementary quantized fields has led to many attempts to understand quantum gravity. None has been completely successful.

Nevertheless, a number of illuminating insights concerning the structure of quantum gravity have been obtained using Feynman's path integral approach to quantization (see, for instance, [1]). The path integral has the well-known property that it can be evaluated unambiguously only for Euclidean imaginary time, although the results are continued back to the Minkowski regime. Approxima-

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tions to the path integral quantization for a theory can therefore be developed first by examining classical Euclidean solutions of the theory and then making a systematic perturbative expansion around these solutions. Since the weight of a given path in the path integral is proportional to the exponential of minus the action, the minimum-action Euclidean solution may dominate the path integral. Quantum amplitudes found by expanding around such a solution have a good chance of being fairly accurate.

§(2): *Instantons*

One example of a minimum-action solution to a nonlinear field theory which has recently attracted a great deal of interest is the *instanton* solution to the Yang-Mills equations found by Belavin, Polyakov, Schwarz, and Tyupkin [2]. This solution is called an instanton because its Yang-Mills field strength is concentrated around one point in four-dimensional Euclidean space-time. In the distant past and distant future, the field strength vanishes: a bump in the field strength appears for an instant of time, then dies away.

The instanton solution arises in an intriguing fashion. First one examines the Yang-Mills equations

$$\partial_\mu F_{\mu\nu} + [A_\mu, F_{\mu\nu}] = 0 \tag{2.1}$$

and the identity

$$\epsilon_{\mu\alpha\beta\gamma}(\partial_\alpha F_{\beta\gamma} + [A_\alpha, F_{\beta\gamma}]) = 0 \tag{2.2}$$

which follows from the definition of the field strength

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$$

Then one observes that if $F_{\mu\nu}$ is self-dual or anti-self-dual,

$$F_{\mu\nu} = \pm\widetilde{F}_{\mu\nu} \equiv \pm \tfrac{1}{2}\, \epsilon_{\mu\nu\alpha\beta} F_{\alpha\beta} \tag{2.3}$$

the identity (2.2) implies that the Yang-Mills equations (2.1) are satisfied. Furthermore, one can show that field strengths satisfying equation (2.3) are absolute minima of the Yang-Mills action.

Now we notice that equation (2.3) contains only first derivatives of the potential A_μ, and yet it implies the solution of equation (2.1), which contains second derivatives of A_μ. Thus in this special case it is possible to reduce the Yang-Mills equations to a first-order differential equation. Suppose one chooses the Ansatz

$$A_\mu = \rho(r) g^{-1} \frac{\partial g}{\partial x_\mu} \tag{2.4}$$

for the $SU(2)$ Yang-Mills gauge potential, where $r^2 = t^2 + \mathbf{x}^2$, $g = (t - i\boldsymbol{\tau}\cdot\mathbf{x})/r$ and $\{\boldsymbol{\tau}\}$ are the Pauli matrices. Then if we require the field strength to be self-dual, $\bar{F}_{\mu\nu} = \tilde{F}_{\mu\nu}$, we find the first-order differential equation

$$\frac{d\rho(r)}{dr} + \frac{2}{r}(\rho - 1)\rho = 0 \tag{2.5}$$

with the solution

$$\rho(r) = r^2/(r^2 + a^2) \tag{2.6}$$

The potential A_μ given by equations (2.4) and (2.6) is the instanton solution to the Yang–Mills equations, which produces a self-dual field strength $F_{\mu\nu}$ concentrated at $r = 0$ and falling off in all directions like $1/r^4$.

The physical implications of the instanton are profound. It provides a new nonperturbative starting point for the path-integral quantization of Yang–Mills field theory. Perhaps the most striking consequence of the instanton is the fact that the Yang–Mills vacuum has an infinitely periodic structure: the true vacuum is a superposition of an infinite number of equivalent vacuum states [3]. The instanton action gives a first approximation to the tunneling amplitude between two adjacent vacuum states. Since instantons carry nontrivial topological quantum numbers, these tunneling amplitudes break various symmetries of the theory. In particular there exists a zero-energy solution to the Dirac equation in the instanton field and this causes a nonconservation of the ninth axial vector current and the breakdown of chiral $U(1)$ symmetry [4]. Instantons also possibly generate CP violation and baryon-number nonconservation.

The existence of spin-$\frac{1}{2}$ zero mode in the instanton field is a consequence of a mathematical theorem, the Atiyah–Singer index theorem [5], and is ultimately traceable to the deep relationship between Yang–Mills theory and the differential geometry of fiber bundles (see, for instance, [6]).

§(3): *Self-Dual Solutions of Euclidean Gravity*

The fact that Yang-Mills theory and its instanton solutions are intimately related to geometry suggests that we examine the other major link between geometry and physics, i.e., Einstein's theory of gravitation. By exploring a number of parallels between Yang-Mills equations and Einstein equations, the present authors were able to discover a new solution of Euclidean gravity [7] which strongly resembles the Yang-Mills instanton.

The gravitational instanton can be derived in the following way: First, consider a Euclidean metric $g_{\mu\nu}$ decomposed into vierbein one-forms $e^a = e^a{}_\mu \, dx^\mu$ as

$$ds^2 = g_{\mu\nu}\, dx^\mu\, dx^\nu = \sum_{a=0}^{3} (e^a)^2 \tag{3.1}$$

and recall that the spin connection one-form ω_{ab} is determined by

$$de^a + \omega_{ab} \wedge e^b = 0, \qquad \omega_{ab} = -\omega_{ba} \tag{3.2}$$

Then note that self-duality (or anti-self-duality) of the spin connection one-form,

$$\omega_{ab} = \pm\tilde{\omega}_{ab} \equiv \pm\tfrac{1}{2}\,\epsilon_{abcd}\,\omega_{cd} \tag{3.3}$$

implies self-duality of the curvature two-form,

$$R_{ab} \equiv d\omega_{ab} + \omega_{ac} \wedge \omega_{cb} \equiv \tfrac{1}{2}\,R_{abcd}\,e^c \wedge e^d \tag{3.4}$$

But self-duality of the curvature two-form,

$$R_{ab} = \pm\tilde{R}_{ab} \equiv \pm\tfrac{1}{2}\,\epsilon_{abcd}R_{cd} \tag{3.5}$$

together with the cyclic identity $\epsilon_{ebcd}R_{abcd} = 0$ imply that the empty space Einstein equations are satisfied. Thus equation (3.5) is the gravitational analog of the fact that self-dual Yang-Mills field strengths satisfy the Yang-Mills equations.

Now we observe that equation (3.3) contains only first derivatives of the metric while implying the validity of equation (3.5) and hence Einstein's equations. This means that we will be able to reduce Einstein's equations to a first-order differential equation, analogous to the Yang-Mills case.

All that is needed now is a good guess for the metric to play the role of equation (2.4). An obvious tactic is to study the flat metric in four-dimensional polar coordinates,

$$ds^2 = dr^2 + r^2(\sigma_x{}^2 + \sigma_y{}^2 + \sigma_z{}^2)$$

where

$$\begin{aligned} \sigma_x &= \tfrac{1}{2}(\sin\psi\, d\theta - \sin\theta\,\cos\psi\, d\phi) \\ \sigma_y &= \tfrac{1}{2}(-\cos\psi\, d\theta - \sin\theta\,\sin\psi\, d\phi) \\ \sigma_z &= \tfrac{1}{2}(d\psi + \cos\theta\, d\phi) \\ 0 \leqslant \theta < \pi, &\quad 0 \leqslant \phi < 2\pi, \quad 0 \leqslant \psi < 4\pi \end{aligned} \tag{3.6}$$

We then examine the following modification of the flat metric:

$$ds^2 = f^2(r)\,dr^2 + r^2[\sigma_x{}^2 + \sigma_y{}^2 + g^2(r)\,\sigma_z{}^2] \tag{3.7}$$

A straightforward calculation shows that the spin connections ω_{ab} are anti-self-dual if $f(r)$ and $g(r)$ obey the first-order differential equations

$$\begin{aligned} fg &= 1 \\ g + r\frac{dg}{dr} &= f(2 - g^2) \end{aligned} \tag{3.8}$$

The solution of these equations is

$$g^2(r) = f^{-2}(r) = 1 - (a/r)^4 \tag{3.9}$$

Thus the metric [7]

$$ds^2 = [1 - (a/r)^4]^{-1} dr^2 + r^2 \{\sigma_x{}^2 + \sigma_y{}^2 + [1 - (a/r)^4] \sigma_z{}^2\} \tag{3.10}$$

satisfies the Einstein equations with anti-self-dual curvature.

One must of course check that the manifold defined by the above metric is geodesically complete. One finds that all is well, provided the range of ψ is changed to [8]

$$0 \leqslant \psi < 2\pi \tag{3.11}$$

This means that at ∞, the coordinate x_μ is identified with its TP conjugate partner $-x_\mu$. Except for this identification, the metric approaches a *flat-space* metric at ∞. The manifold's natural origin is at $r = a$, where the metric is in fact regular and there is an instantonlike bump in the curvature.

The new gravitational instanton metric (3.10) is now known to be the first of a family of multiple-instanton metrics discovered subsequently by Gibbons and Hawking [9]. These interesting metrics have been shown by Calabi [10] and by Hitchin [11] to arise in a natural mathematical context.

We now turn to the physical meaning of the gravitational instanton metric (3.10). In the first place, since the metric satisfies the Einstein empty-space equations, its scalar curvature and classical action *vanish.* Therefore the gravitational instanton gives a dominant contribution to the path integral as important as the contribution of the flat metric itself. Secondly, since the metric approaches that of a flat space at infinity modulo an identification, it makes contributions to the asymptotic scattering states of conventional quantum field theory and causes a certain type of symmetry violation. Perry [12] has pointed out that the simplest nontrivial amplitude induced by the gravitational instanton occurs in the particle four-point function. He has shown that in the electron-positron scattering process incoming electrons are transformed by an instanton into outgoing positrons with reversed helicity. This phenomenon is the TP reversal which takes place as a particle passes through an instanton field (recall that x_μ had to be identified asymptotically with its TP conjugate $-x_\mu$). We also remark that while the gravitational instanton does not contribute to the zero-frequency modes of a spin-$\frac{1}{2}$ Dirac particle, it does produce zero-frequency solutions of the spin $\frac{3}{2}$ Rarita–Schwinger equation [13] and thus possibly breaks chiral $U(1)$ symmetry.

Thus our gravitational instanton solution (3.10) of the Einstein equations bears a remarkable resemblance to the instanton solution of Yang–Mills theory. We may summarize the parallels in the following table:

Yang-Mills Solution of Reference 2	Einstein Solution of Reference 7
self-dual field strength	(anti)-self-dual curvature
field-strength bump at origin	curvature bump at origin
$A_\mu \rightarrow$ pure gauge at infinity	$g_{\mu\nu} \rightarrow$ locally flat spact at infinity
no singularities	geodesically complete
finite action	zero action
CP-changing amplitudes	TP-changing amplitudes
spin-$\frac{1}{2}$ zero-frequency modes	spin-$\frac{3}{2}$ zero-frequency modes

§(4): *Conclusion*

New insights into the nature of relativistic quantum field theories will be gained by path-integral quantization methods. In particular, the expansion of the path integral around minimum-action solutions will prove to be a powerful tool in the analysis of nonlinear field theories. Just as the instanton solution of Yang-Mills equations has led to a deeper understanding of the theory, the gravitational instanton we have presented here promises to play a similar role in quantum gravity.

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