Misty States for Phase Kickback

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In this document I want to prove the phase kickback phenomenon entirely visually using the (pure) misty state formalism introduced in "Q is for Quantum" by Terry Rudolph. For now it is assumed that the reader is familiar with the diagrams in Part 1 (Q-Computing). At the outset we need to establish some facts which we will need later during the actual proof. We start with the effect of a NOT gate on a superposition of states (with phase) as shown on the left:

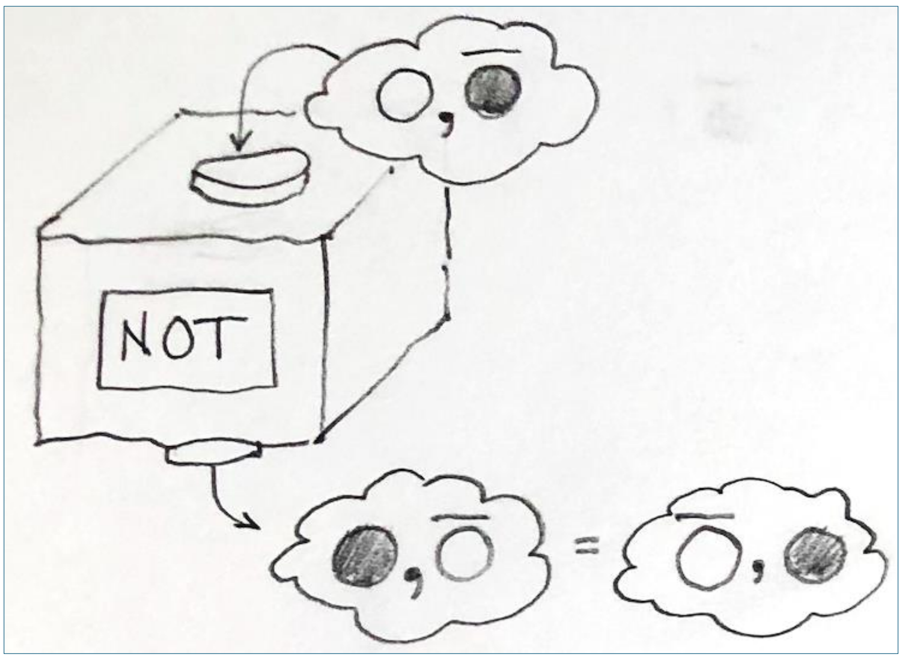


Figure 1. This is $\mathrm{NOT}\left(\frac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right)\right) = -\left(\frac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right)\right)$ or, simpler: $\mathrm{NOT}\left(|-\rangle\right) = -|-\rangle$

We remind the reader that the order is not important in a mist and that NOT is a linear gate, that is, when we apply the gate to a mist of states the result is a mist comprised of NOT applied to each of the states in the original mist. The phase acts as a minus and is also linear.

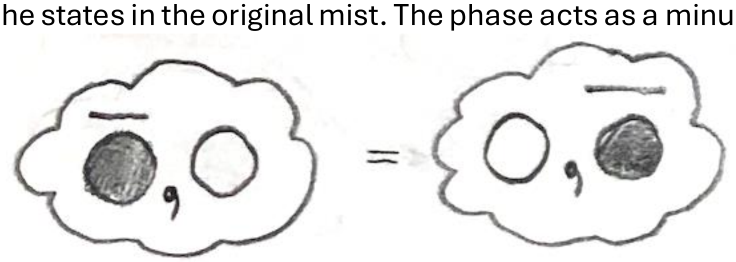


Figure 2. The order of states in a mist is not important, but an order is usually preferred.

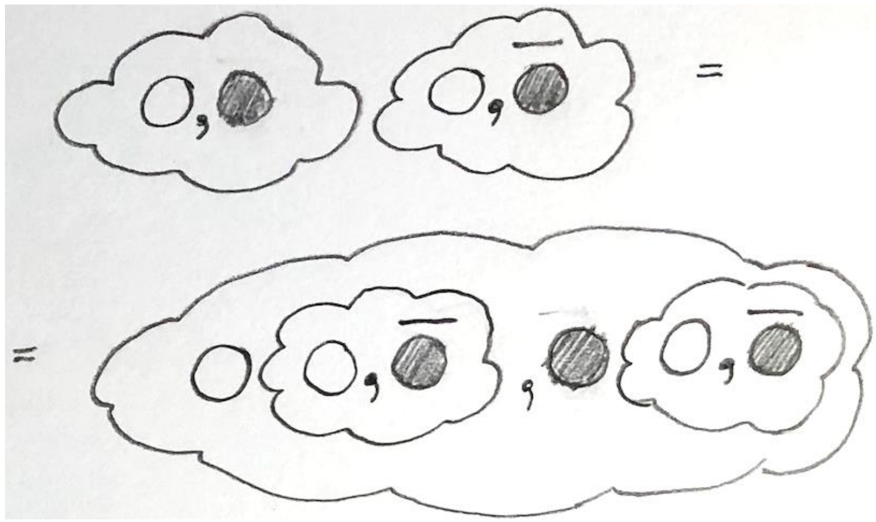


Figure 3. A two qubit state is expressed as a (tensorial) product. The operation is known as FOIL (acronym) in pre-algebra. A mist acts like a sum, while a (tensorial) product as a (non-commutative) multiplication. The phase acts as a sign (plus or minus). A negative phase is shown as an overbar. In the process we may decide to keep the second mist intact.

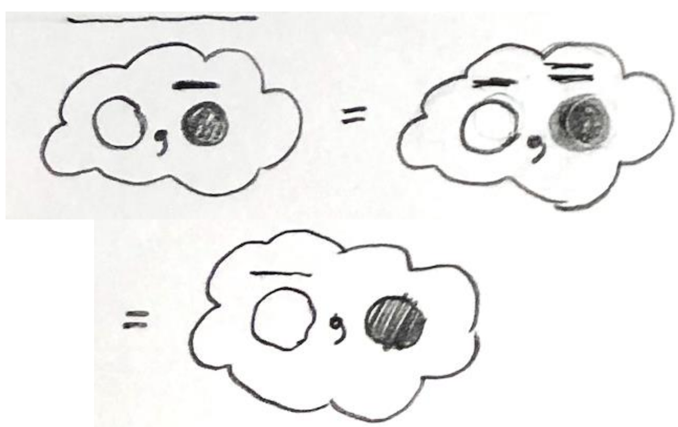


Figure 4. A negative phase applied to a mist distributes over its constituent states.

For every diagram we draw there is a corresponding, unambiguous mathematical translation using Dirac notation but we won't use (or show) that here b/c it might distract the reader from what we want to show. The reader is encouraged now to make sure that they understand the derivation presented in Figure 1 beyond any doubt, as the result will now be used below.

So now let's remind ourselves what we want to prove:

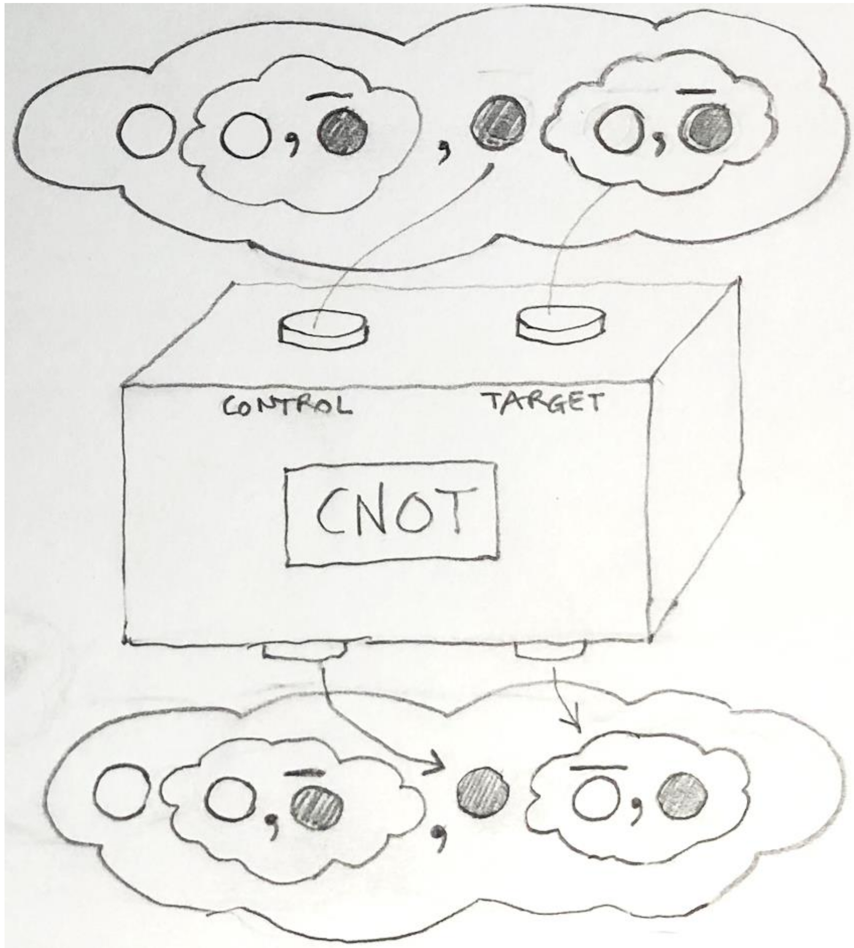


Figure 5. The effect of a controlled NOT gate on the state $|+\rangle|-\rangle$ as shown in Figure 3.

Note that if you trace carefully the transformations in Figure 5 above you can see the effect of the NOT gate on the second mist (qubit) as anticipated earlier in Figure 1.

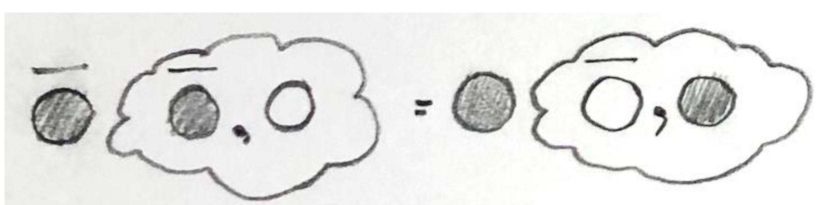


Figure 6. Since a (tensorial) product acts essentially as a multiplication (though without being commutative, so order of qubits does matter) phase acts as expected. Here we move the sign from the first qubit to the mist that represents the second qubit (see also Fig. 4).

So now we're ready to process the output from Figure 5.

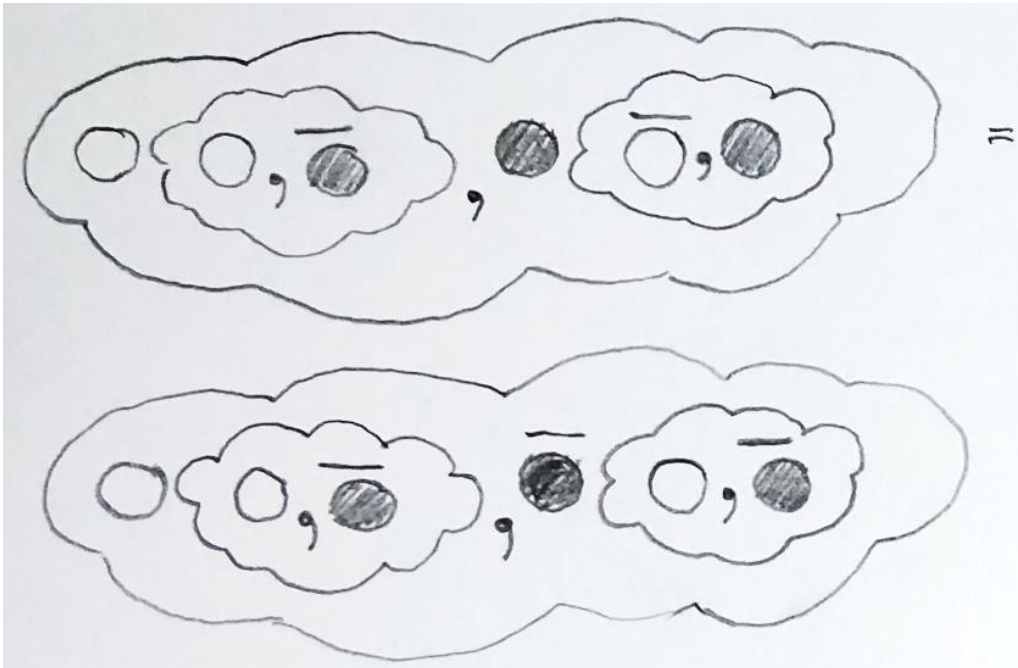
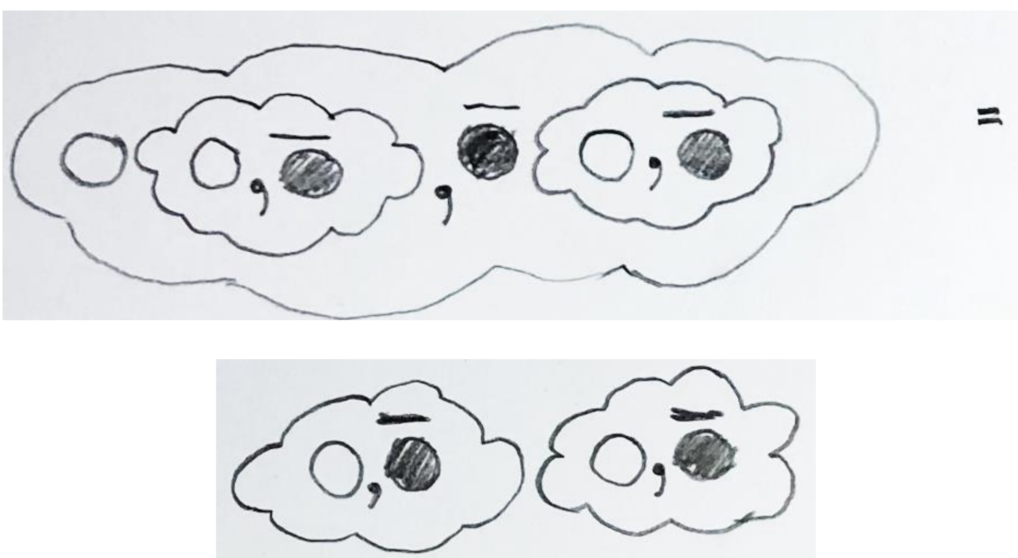


Figure 7. Here's how we rewrite the output from Fig. 5 using the result stated in Fig. 6

And now we can FOIL back by using the second mist as a common factor:



And now we're done.

The summary then of what we did here can be drawn as follows:

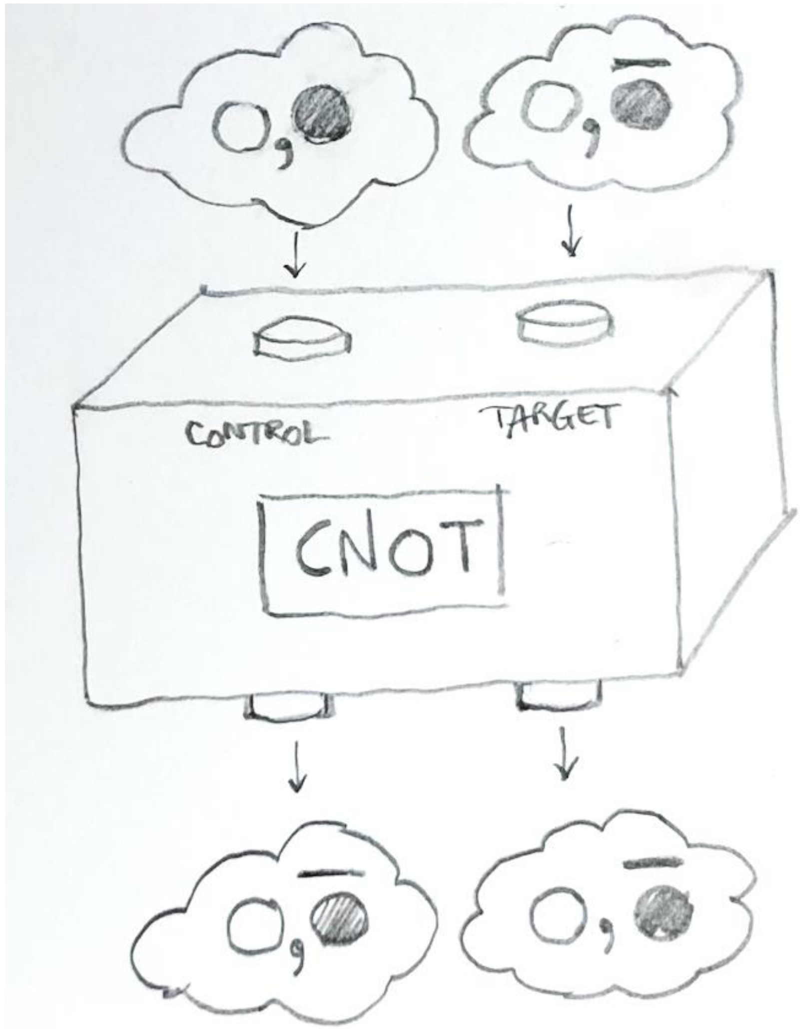


Figure 8. The phase kickback with misty states.

And here's what we proved:

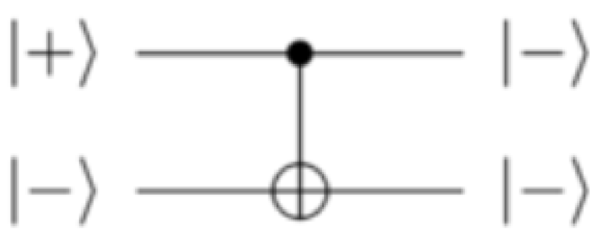


Figure 9. Phase kickback (conventional notation, from Quantum Country).