On the Foundations of Corecursion

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Abstract

We consider foundational questions related to the definition of functions by corecursion. This method is especially suited to functions into the greatest fixed point of some monotone operator, and it is most applicable in the context of non-wellfounded sets. We review the work on the Special Final Coalgebra Theorem of Aczel [NWF] and the Corecursion Theorem of Barwise and Moss [VC]. We prove some results relating the two approaches, and we present a framework from which one can derive the results of both.

1 Introduction

By a *stream of natural numbers* we mean a pair $\langle n, s\rangle$ where $n \in N$ and s is again a stream of natural numbers. Let $f : N \to N$. Consider the following function which purports to define a function from N into the streams:

$$iter_f(n) \quad = \quad \langle n, iter_f f(n)\rangle \tag{1}$$

For each n, $iter_f(n)$ is a stream, so $iter_f$ itself is a function from numbers to streams. This is an example of a function defined by *corecursion*. It seems to be similar to recursion, since the symbol $iter_f$ is used on both sides of (1). On the other hand, there is no "base case," so something different seems to be going on. The purpose of this paper is to consider the foundational problem of justifying such definitions. We review the existing work on this and we offer a general approach as well.

Here is a second example: A *tree of natural numbers* is a triple $\langle n, t_1, t_2\rangle$ where $n \in N$ and t_1, t_2 are again trees of natural numbers. Consider the following function τ from $\{a, b, c\}$ into trees over natural numbers:

$$\begin{array}{lcl} \tau(a) & = & \langle 61, \tau(c), \tau(c)\rangle \\ \tau(b) & = & \langle 4, \tau(b), \tau(c)\rangle \\ \tau(c) & = & \langle 4, \tau(a), \tau(c)\rangle \end{array} \tag{2}$$

This again is a corecursion, this time into the trees. Our main foundational aim is to offer a general theory of such definitions, modeled after what is now standard for definition by recursion. The approach should be broad enough to cover (1) and (2), as well as any "similar" example that one would expect to arise.

Our last example has to do with functions into the universe of sets. We ask for a function h defined on $\{0, 1\}$ so that

$$\begin{array}{lcl} h(0) & = & \{3, \{h(1)\}\} \\ h(1) & = & \{4, h(0), h(1)\} \end{array} \tag{3}$$

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Someone familiar with set theory would be quick to notice that this last example calls for sets which are not wellfounded (and which therefore do not exist according to *ZFC*, the usual axioms of set theory). Someone working in *ZFC* could with confidence say that there is no function h as in (3). But this reasoning denies that (1) and (2) have solutions. And there are many fields where people *do* want solutions to equations similar to those. (For example, systems of stream equations arise in work on computer hardware.)

In order to have solutions to these corecursion equations, our work takes place in the universe of *non-wellfounded sets* (also called *hypersets*) introduced by Forti and Honsell and also by Aczel. This is an extension of the more standard universe of wellfounded sets, obtained by adopting the Anti-Foundation Axiom (*AFA*) in place of the Foundation Axiom. The observation that *AFA* could give a foundation for corecursions is due to Aczel (see [NWF]). Indeed, his work on *AFA* originated in a study of models of calculi for concurrency where one wants corecursion.

Foundations of Recursion To get a feeling for the kind of results we are after, consider the following well-known results pertaining to recursion:

(A) Let $f : N \times N \to N$, and let $a \in N$. Then there is a unique function $g : N \to N$ so that $g(0) = a$ and for all n, $g(n+1) = f(g(n), n)$.

(B) $(N, 0, s)$ is initial in the category of Peano systems.

(C) Let $(W, <)$ be a wellorder. For each $w \in W$, write seg_w for the set $\{v \in W : v < w\}$. Let $f : Sets \times W \to Sets$. There is a unique function g defined on W so that for all $w \in W$, $g(w) = f(g \upharpoonright seg_w, w)$.

(D) Let P be a poset in which every subset has a supremum. Then every monotone $f : P \to P$ has a fixed point.

Of these statements, (A) surely justifies most recursions that most mathematicians will ever have to consider. On the other hand, when we consider other important kinds of recursion (say, recursion on finite binary trees, on sentences, or on proofs), then we are lead to more general forms. Point (B) is a more abstract formulation. While not more generally applicable than (A), it suggests analogies that one could pursue to get more general approaches to recursion. Statement (C) is a very general statement which justifies practically all recursions. (However, (C) does not do everything: it does not cover the case when W is a proper class like the class of all ordinals. As rarefied as this point may seem, it is not entirely foreign to the concerns of this paper.) To understand (C) from first principles would take more work than (A) or (B), since one would have to know something about wellorders; to prove it would take even more since the Replacement Axioms of set theory are needed. In fact, one way to motivate a certain part of elementary set theory is as the search for axioms and definitions that allow us to prove something like (C). Note also that (B) and (C) take off in different directions from (A), as does (D). This last statement implies (C) and is therefore more abstract.

Now what we have in mind for corecursion are statements like (B) and especially (C). In the category-theoretic direction, there are connections of *finality* with corecursion from [NWF] that we review in Section 2.1. Closer to (C), the work of [VC] brings in a conceptual apparatus that also justifies many corecursions. We cover this in Section 2.4. That work has an interesting relationship to the category-theoretic approach, and indeed the whole discussion of this paper raises some apparently new questions about the category of sets. Although the paper can be read without consulting [VC], we have generally omitted the longer proofs which do appear there. Also, those not familiar with non-wellfounded sets would probably need to consult either [NWF] or [VC]. Section 3 presents a blend of the two approaches explored earlier in the paper.

But is it recursion? When dealing with corecursion, an immediate concern is the extent to which it can be reduced to ordinary recursion. Let's take streams for example; for concreteness, we'll focus on streams over the natural numbers. Streams are most naturally viewed as ordered pairs, each consisting of a natural number and another stream. However, the set of streams can also be modeled (though not as naturally) as the set of functions $s : N \to N$. Likewise, functions on or into streams can be modeled in this way. For example, the iteration function defined by (1), which defines a stream by iteration of a function $f : N \to N$ on a given input, can be modeled as follows. First, define by recursion on N the two-place function g: $g(0, n) = n$, and $g(m+1, n) = f(g(m, n))$. Now set $iter_f(n)(m) = g(m, n)$. Recall that we are modeling streams as functions, so a function from N into streams should be modeled as a function from N into functions on N. So by choosing an appropriate model, we have reduced this corecursive definition into one defined by the more comfortable recursion on natural numbers.

A valid question at this point is: given that we can make this reduction, is the value of the "most natural" model of streams worth the extra effort that it involves (i.e., non-wellfounded set theory)? While this is not a question we are proposing as a straw man to be knocked down, a partial response is as follows. We have come up with a reasonable model of streams, if not the most natural one, within the context of wellfounded set theory. But will this always be possible? For example, what if one needs to define functions in the style of the function defined by equations (3)? There is certainly no obvious way in which to "reasonably" model this in a wellfounded set theory. And there is always the concern of a uniform method; is there a single result that will allow us to reduce any corecursive definition to a recursive one?

We show in this paper that it is possible in a wide variety of cases to reduce corecursion to recursion. This is not entirely obvious, and the issues surrounding the reframing are interesting. It is possible to make a reduction in many (and perhaps all) of the cases of current interest. We discuss this in Section 4. However, it is an open question whether such a reduction is possible in every case.

2 Previous work on corecursion

In this section, we survey work on corecursion from [NWF] and [VC]. Our goals are not to give all the proofs but rather to motivate the whole technical machinery that has been introduced. While similar, the approaches differ in that Aczel's framework deals with endofunctors on the category of classes, while Barwise and Moss work with operations on sets that are not assumed to preserve any kind of structure. Although the two approaches feel similar, it is not immediately clear what the relation is between them. In Section 2.5, we present some results relating the two approaches. In Section 3 we offer a framework from which one can derive the main results of both approaches at the same time.

2.1 Endofunctors on Class

An *operator* is an operation , taking sets to sets. , is *monotone* if for all sets $a \subseteq b$, $, a \subseteq , b$. A monotone operator extends in a natural way to classes by taking

$$, C \quad = \quad \bigcup\{, a \mid a \subseteq C\}.$$

The resulting operator on classes is again monotone, and it is *set-based*: if $a \in , C$, then there is a set $b \subseteq C$ such that $a \in , b$.

Example 2.1 The kinds of examples we have in mind in this paper are operations like $a \mapsto a$, $a \mapsto \mathcal{P}(a)$, $a \mapsto A \times a$ for a fixed set A, $a \mapsto A \to a$, etc. Set theoretic operations which are not monotone include $a \mapsto \{a\}$ and $a \mapsto a \to a$. The composition of monotone operators is monotone.

Theorem 2.1 *If , is an operator on classes which is monotone and set-based, then , has a least fixed point* $,_*$ *and a greatest fixed point* $,^*$.

The assertion about least fixed points is a well-known consequence of definition by recursion on the ordinals. The greatest fixed point result is due to Aczel [NWF]. Indeed, there is a nice representation for the class $,^*$:

$$,^* \quad = \quad \bigcup\{b \mid \; b \text{ is a set and } b \subseteq , (b)\}. \tag{4}$$

Aczel [NWF] also proved basic results relating these fixed points to initial algebras and final coalgebras for certain endofunctors. To state these, let Class be the category of classes whose morphisms are the set-based operations. Now most of the usual operations of set theory are the object parts of endofunctors on the category Set (and hence on Class). Some exceptions to this are $a \mapsto \{a\}$ and $a \mapsto \bigcup a$. But these are not the kind of operations on sets which are pertinent to a discussion of corecursion. All of the examples of monotone operations from Example 2.1 do extend to functors. And those functors have the following important property as well:

Definition An endofunctor , : Class $\to$ Class is *standard* if its object part is monotone and set-based, and if its morphism part preserves inclusion maps between classes. That is, if $i(C, D)$ is the inclusion of C into D, then $,_{i(C,D)} = i_{\Gamma(C),\Gamma(D)}$.

We know that , has a least fixed point $,_*$. We get an algebra for , , also denoted $,_*$, by considering $(,_*, i(,_*, ,_*))$. Similarly, the greatest fixed point $,^*$ may be regarded as an algebra or a coalgebra for , .

Theorem 2.2 (Aczel [NWF]) *Let , be a standard endofunctor on* Class*. The algebra* $,_*$ *is initial in the category of algebras for , .*

A closely related result appears below in Theorem 3.1. Incidentally, although we believe that some sort of of requirement such as standardness is needed to get initiality of the least fixed point algebra, we do not have any concrete examples.

For final algebras the natural guess is that $,^*$ would turn out to be the final coalgebra. However, this is not correct. First, consider the identity functor $, a = a$. The greatest fixed point is the universe of sets. But this is clearly not the final coalgebra. For this reason, we usually impose the following condition:

Definition , is *proper* if for all sets a, $, (a) \subseteq V_{afa}[\mathcal{U}]$.

All of the examples at the beginning of this paper may be cast in terms of functors which are proper.

However, even assuming properness, we still cannot be sure that $,^*$ is the final coalgebra. To see why, consider the functor $, a = B \times a$, where the cartesian product is defined from the usual Kuratowski pair, where $\langle p, q \rangle = \{\{p\}, \{p, q\}\}$; B is an arbitrary fixed set. , works on morphisms in the natural way. Unlike the identity, this is a functor of interest for corecursion. Assuming the Foundation Axiom, the only fixed point of , is the empty set. Hence $,^* = \emptyset$. But $\emptyset$ is not a final coalgebra for , . Even worse, consider the case of the identity functor $, (a) = a$. Then a final coalgebra would be a terminal object, hence a singleton set. But the greatest fixed point is the universe of all sets.

These examples shows that the matter of final coalgebras is tied up with both with the Foundation Axiom and with additional constraints that the endofunctor must satisfy. So to go further, we'll need to make a digression on matters related to the Anti-Foundation Axiom.

But before we do that, we should explain why final coalgebras are related at all to the matter of corecursion. The point is that the example corecursions (1)–(3) can be understood as instances of final coalgebra results. For example, look back at (2). Here the operator is $, (a) = N \times a \times a$, made into a functor in the obvious way. Let Tr be the greatest fixed point of , . Let's assume for a moment that Tr is also the final coalgebra of , . The equations (2) define a coalgebra $(\{a, b, c\}, \pi)$, where

$$\pi : \{a, b, c\} \rightarrow N \times \{a, b, c\}$$

is given by $\pi(a) = \langle 61, c, c \rangle$, $\pi(b) = \langle 4, b, c \rangle$, $\pi(c) = \langle 4, a, c \rangle$. Then by finality, there is a unique map $\tau : \{a, b, c\} \rightarrow Tr$ such that $\pi \circ , \tau = \tau$. Tracing through the definitions, we get that τ satisfies (2). Moreover, the uniqueness part of finality insures that τ is unique. In this way, all of the corecursions of interest are tied up to final coalgebra theorems.

2.2 Background on the Anti-Foundation Axiom

In order to state and work with *AFA*, it is most convenient to work with a set theory that has more than just sets. We also want to have "urelements" (called "atoms" in [NWF] and elsewhere) in the universe. These are objects which are not sets because they have no elements, but they can belong to sets. The reason that urelements are not part of the usual set theories is pretty much the same reason why non-wellfounded sets are banished: they are not needed for the foundational work of set theory. However, in applied fields, it is often convenient to think of the set theoretic universe as being built over a collection of urelements, since the coding of the basic objects as sets is at best irrelevant and at worst misleading to applications.

As it happens, adding urelements to set theory is not very hard. One adds a relation symbol $\mathcal{U}$ to the language of set theory. For example, we define "x is a set" to mean that $\neg\mathcal{U}(x)$. We abuse notation a bit and write $x \notin \mathcal{U}$ in this case. We take an axiom that says that urelements do not themselves have elements. The usual axioms need to be relativized to sets. We also need an existence assumption for urelements. One way to do this is to add a two-place function symbol new to the language of set theory and take the following axiom:

STRONG AXIOM OF PLENITUDE Concerning the operation new:

1. For all sets a and all sets $b \subseteq \mathcal{U}$, $\mathsf{new}(a, b) \in \mathcal{U} - b$.
2. For all $a \neq a'$ and all sets $b \subseteq \mathcal{U}$, $\mathsf{new}(a, b) \neq \mathsf{new}(a', b)$.

The idea is that $\mathsf{new}(a,b)$ gives an urelement which is new in the sense that it does not belong to b. And for fixed b, the operation of giving new urelements for b is injective.

We use letters like x, y, and z for urelements, X,Y, Z, etc. for sets of urelements, a, b, c, etc. for sets, p, q, r, etc. for objects which can be either sets or urelements, and C, D etc. for classes (including the sets). We usually prefer to write $\mathsf{new}_b(a)$ instead of $\mathsf{new}(a,b)$.

Definition For each set a, $TC(a)$ is the smallest set including a (as a subset) and which is transitive. (This means that if $p \in q \in TC(a)$, then $p \in TC(a)$.) The *support of* a, $support(a)$ is $TC(a) \cap \mathcal{U}$. The elements of $support(a)$ are the urelements which are "somehow involved" in a. A set a is *pure* if $support(a) = \emptyset$. Finally, for all sets or classes $A \subseteq \mathcal{U}$, we define

$$V_{afa}[A] \quad = \quad \{a \mid a \text{ is a set and } support(a) \subseteq A\}.$$

This is always a proper class, of course. If $A = \emptyset$, we omit it from the notation and just write V_{afa}. So V_{afa} is the class of all pure sets.

Definition A *substitution-like operation* is a function

$$F : V_{afa}[\mathcal{U}] \cup \mathcal{U} \to V_{afa}[\mathcal{U}] \cup \mathcal{U}$$

with the property that for all sets a,

$$F(a) \quad = \quad \{F(p) \mid p \in a \text{ is a set or urelement}\}$$

It is easy to check that the substitution-like operators are closed under composition. Of course, the basic idea behind substitution-like operators is that they work "recursively." The action of such an operator on a given set is determined by "descending down $\in$-chains", stopping only at urelements. Of course, in a set theory without the Foundation axiom, these $\in$-chains need not terminate; assuming the Anti-Foundation Axiom there are some which do not. With *AFA*, these "recursions" have no base case. For this reason, special results pertaining to substitution-like operations (such as well-definedness) must either be taken as axioms or proven. Indeed, the same problem happens with corecursion, and this is the overall contribution of this paper.

Definition A *substitution* is a function s whose domain is a set or class of urelements. Further, s is *proper* if its domain is a set X, and for all $x \in X$, $s(x)$ is a set of sets.

For any set $X \subseteq \mathcal{U}$, write id_X for the identity map on X.

Now *AFA* is the conjunction of the following two assertions:

1. Every substitution s has a unique extension $[s]$ to a substitution-like operation satisfying the condition that

$$[s]p \quad = \quad \begin{cases} s(p) & \text{if } p \in dom(s) \\ p & \text{if } p \in \mathcal{U} - dom(s) \\ \{[s]p \mid p \in a\} & \text{if } p \text{ is a set} \end{cases}$$

2. For every proper substitution $e : X \to \mathcal{P}(V_{afa}[X \cup Y])$ there is a unique substitution $s : X \to \mathcal{P}(V_{afa}[Y])$ so that for all $x \in X$, $s(x) = [s](e(x))$. This substitution s is called the *solution to* e.

, (a)	$,_*$	$,^*$ (assuming FA)	$,^*$ (assuming AFA)
a	$\emptyset$	class of all sets	class of all sets
$\mathcal{P}(a)$	V_{wf}	V_{wf}	V_{afa}
$A \times a$	$\emptyset$	$\emptyset$	infinite streams over A
$A \times a \times a$	$\emptyset$	$\emptyset$	infinite binary trees over A
$(A \times a) \cup \{\epsilon\}$	$\mathit{FinSeq}(A, \epsilon)$	$\mathit{FinSeq}(A, \epsilon)$	$\mathit{FinSeq}(A, \epsilon)$ + infinite streams over A
$\mathcal{P}(A \cup a)$	$V_{wf}[A]$	$V_{wf}[A]$	$V_{afa}[A]$
$\mathcal{P}(A \times a)$	$\emptyset$	$\emptyset$	canonical non-deterministic automata over A

Table 1: Least and greatest fixed points of some monotone operators

This is not the usual way to state AFA, but it is equivalent. Henceforth in this paper, we adopt AFA; indeed AFA is needed for most if not all of the results to come. AFA has a number of consequences. For example, it implies that many of our example operators have greatest fixed points which are themselves interesting. Some of these are listed in Table 1. The greatest fixed points exist without assuming AFA.

A few brief points about the entries in the table: Assuming FA makes the least and greatest fixed points the same in all cases except the identity. Assuming FA often trivializes the fixed points, as can be seen from the $\emptyset$ entries. Assuming AFA, the greatest fixed point is strictly larger than the least in all cases.

In the second line of the table, V_{wf} is the class of wellfounded sets.

The "infinite" streams over A are the streams with which we began the paper. They are pairs and hence finite sets, but we call them infinite to distinguish them from the fixed points of the operator , $a = (A \times a) \cup \{\epsilon\}$. Concerning that operator, the set $\mathit{FinSeq}(A, \epsilon)$ is the set of finite nested pairs of the form:

$$\langle a_1, \langle a_2, \ldots, \langle a_n, \epsilon \rangle \rangle \rangle$$

where $a_1, \ldots, a_n \in A$. We can think of them as finite sequences from A. The least fixed point of this operator gives the set $\mathit{FinSeq}(A, \epsilon)$, and the greatest gives the same set together with the infinite streams over A (the greatest fixed point of , $a = A \times a$.) This bigger set might be used to model possibly-terminating streams.

In the operator , $a = \mathcal{P}(A \cup a)$, A is a set of urelements.

In the final operator, , $a = \mathcal{P}(A \times a)$, we think of A as a set of actions of some automaton. Think of the elements of $,^*$ as states of an automaton. These states are then sets of pairs, each pair being an action and then another state. So each state is exactly the transition relation it determines; no isomorphism is needed. For this reason, the states are called *canonical*. The connection of AFA to the general matter of canonical objects is discussed in [VC].

2.3 Corecursion via the Special Final Coalgebra Theorem

We can now return to the discussion of final coalgebras for endofunctors on Class. As we saw above, a result relating greatest fixed points to final coalgebras must involve some condition on the functor. We review Aczel's condition of *uniformity on maps*.

Definition , is *uniform on maps* if for every class A of pure sets, there is a set $\bar{A} \subseteq \mathcal{U}$, a bijection $den : X \to A$, and a map $c : , A \to V_{afa}[X]$ such that for every function $f : A \to B$,

$$, f = [f \circ den] \circ c.$$

The import of this condition is that $(, f)a$ is determined by substitution: it is $[f \circ den]c(a)$. The name *den* stands for "denotation"; the idea is that $f \circ den$ is defined on X. The idea is that urelements are somehow like variable sets, so the content of f is captured by $f \circ den$. The advantage of $f \circ den$ is that it is a substitution. So we can apply $[f \circ den]$ to $c(a)$.

Let's see how this condition of uniformity on maps is verified with the functor $, (a) = B \times a$, where B is a fixed set. For the time being, we assume that B is a set of pure sets; some of our later discussion will turn on this point. Of course, , acts on maps in the obvious way, so that given $f : a \to a'$, $(, f)(b, d) = \langle x, fd \rangle$ for all $d \in a$ and $b \in B$. We apply the definition with $A = \{0, 1\}$. For X we take any set $\{u_0, u_1\} \subseteq \mathcal{U}$. Let *den* be given by $den(i) = u_i$. For the map c we take $\langle b, i \rangle \mapsto \langle b, u_i \rangle$. As required, c maps into $V_{afa}[X]$. For any $f : A \to A'$,

$$\begin{array}{rcl} , f(\langle b, i \rangle) & = & \langle b, fi \rangle \\ & = & [f \circ den]\langle b, u_i \rangle \\ & = & [f \circ den]c(\langle b, i \rangle) \end{array} \qquad (5)$$

This is exactly what we want.

Theorem 2.3 (Special Final Coalgebra Theorem [NWF]) *Assume that , is standard and also uniform on maps. The coalgebra* $,^*$ *is final in the category of coalgebras for , .*

We might mention that Aczel calls this result the Special Final Coalgebra Theorem in order to differentiate it from a more general result (see Aczel [NWF] and also Aczel and Mendler [AM]) on final coalgebras for endofunctors on Class. The general result is stronger since it does not use *AFA* in its hypotheses, but consequently it does not give a connection to greatest fixed points. Such a connection is essential to the work of this paper.

Theorem 2.4 (Corecursion for Standard Endofunctors that are Uniform on Maps) *Let , be a standard endofunctor that is uniform on maps,* $f : A \to , A$ *any coalgebra for , . Then there is* $Z \in V_{afa}[X]$, $c : , A \to Z$, *and a unique* $\varphi : A \to ,^*$ *such that*

$$\varphi \quad = \quad [\varphi \circ den] \circ c \circ f.$$

Proof By the Special Final Coalgebra Theorem, there is a unique map $\varphi : A \to ,^*$ such that $\varphi = (, \varphi) \circ f$. Since , is uniform on maps, $, \varphi = [\varphi \circ den] \circ c$. The uniqueness of φ is shown the same way. ⊣

We discuss an example of a related result in the next section; see Example 2.3.

Before we go on, let's re-examine the functor $, a = B \times a$, but dropping the assumption that B is a pure set. In this case, we have a problem concerning X: if $support(B) \cap X \neq \emptyset$, then $[f \circ den]$ might not be the identity on B. (Note that we used the condition that B is pure when we calculated in (5) that $[f \circ den]b = b$. So it is these calculations which would not go through.) In this case we must choose X to be disjoint from the support of B; since B is a set, this is not a problem. We mention this because this is a leading idea for the definitions in the next section.

A second problem is that c will not map into $V_{afa}[X]$ in this case; the best we can say is $V_{afa}[X \cup support(B)]$. This is not a critical point, so we will largely ignore it.

We feel that the condition of uniformity on maps in [NWF] has an intuitive motivation and at the same time permits one to prove the Special Final Coalgebra Theorem. We prefer to revise the condition to be even more natural, and then later in this paper we will prove a Final Coalgebra Theorem for the new definition. Since the conditions are similar, we give them similar names. Before turning to the definition, we need one more general piece of notation. If s is a substitution and a is any set, then

$$[s]_a \quad = \quad \{\langle b, [s]b\rangle \mid b \in a \text{ is a set.}\}.$$

Thus, $[s]_a$ is a surjective map: $[s]_a : a \to [s]a$. For an example of how this is used, we can restate AFA as follows: for every substitution $e : X \to \mathcal{P}V_{afa}[\mathcal{U}]$ whose domain is a set of urelements there is a unique substitution s with domain X such that $s = [s]_{[e]X} \circ e$.

Definition An endofunctor , on Class or Set is *map uniform* if there is a class $C \subseteq \mathcal{U}$ which is the complement of a set such that whenever $s : X \to b$ is a substitution and $X \subseteq C$, then $, s = [s]_{\Gamma X}$.

Although map uniformity is not the same as that of uniformity on maps, there are some important similarities. First, we mention the motivation for map uniformity. Suppose that s is a substitution with domain X, and suppose that X "had nothing to do" with , . (For example, if , was the functor $a \mapsto B \times a$, then X would have nothing to do with , if $X \cap support(B) = \emptyset$.) Suppose that we wanted to calculate $, s$. The only choice we seem to have is to take the substitution operation $[s]$ and restrict to $, X$. This is exactly what map uniformity has, and this overall point is how we motivate that definition.

Incidentally, the requirement that C contain all but a set of urelements is not really needed: all our results go through if C is just a proper class. We have made the definition this way to facilitate comparison with the approach of the next section.

Second, we have a technical connection between uniformity on maps and map uniformity. The condition in the proposition below is just like uniformity on maps except that we drop the condition on the codomain of c, and we specify c exactly.

Proposition 2.5 *If , is map uniform, then , has the following property: for every class A there is a bijection $den : X \to A$ with $X \subseteq C$ such that for all $f : A \to B$, $, f = [f \circ den]_{\Gamma X} \circ [den]^{-1}_{\Gamma X}$.*

Proof Since C is a proper class, we can find a bijection $den : X \to A$ with $X \subseteq C$. By map uniformity $, den = [den]_{\Gamma X}$ and $, (f \circ den) = [f \circ den]_{\Gamma X}$. By functoriality,

$$, f \quad = \quad , (f \circ den) \circ (, den)^{-1} \quad = \quad [f \circ den]_{\Gamma X} \circ [den]^{-1}_{\Gamma X}.$$

(Note that we used the fact that as a functor, , preserves the bijectivity of maps to get an inverse to $, den$.) ⊣

Theorem 2.6 *Let , be an endofunctor on* Class *which is map uniform and proper. Then $,^*$ is final in the category of , -coalgebras.*

We'll give a proof in Section 2.5, deriving this result from a finality theorem that also implies the corecursion theorem of the next section.

Our overall conclusion on uniformity on maps vs. map uniformity is that while the former seems to be slightly weaker (we do not know for sure), the latter condition is motivated the same way, holds for all the known examples, and also satisfies a finality theorem for greatest fixed points.

2.4 Corecursion via smooth operators

In this section, we review the work on corecursion from [VC]. This work differs from the last section in that we forget about categories and work with directly with monotone operators on sets. However, we also need to make assumptions about the operators involved.

To grasp the definitions to come, let's return to the corecursion theorem of the previous section. Part of what drives the proof of the theorem is that we make use of a "notation system" *den* for A, where $\pi : A \to , A$ is given. The notation system in this case is a set of urelements that "stand in" for A; here, it is the set X. The notation system allows us to define a substitution on urelements which is used to define the final map φ. The requirement that , be uniform on maps allows us to lift an element from , A to a set whose support is contained in X, and hence on which the substitution from X can be applied. It is natural to expect that the set Z in the statement of the Corecursion Theorem can be replaced by , X. Given $den : X \to A$, there are two natural choices of maps from , X to , A: , (den), and $[den]_{\Gamma X}$. The first only makes sense when , is a functor. Since we are interested in the situation in which we may be using operators that are not the set part of functors, we must work with the latter. But now to lift from , X, we require that $[den]_{\Gamma X}$ be injective, so that it can be inverted.

Definition A set $X \subseteq \mathcal{U}$ is *very new for* , if for all substitutions s with domain X, and all sets a,[1]

1. , $(a[s]) =$, $(a)[s]$.
2. If $[s]_a : a \to a[s]$ is injective (and hence bijective), then also $[s]_{\Gamma(a)}$: , $(a) \to$, $(a[s])$ is injective (and hence bijective).

We say that *almost all urelements are very new for* , if there is a *set* $X_\Gamma \subseteq \mathcal{U}$ such that for all sets $Y \subseteq \mathcal{U}$ with $Y \cap X = \emptyset$, Y is very new for $\mathcal{U}$.

, is *smooth* if , is monotone, proper, and if almost all urelements are very new for , .

Example 2.2 All of the main example operators are smooth. For example, consider the operator , $b = A \times b$, where A is a fixed set. Let $X_\Gamma = support(A)$. This is a set, and we claim that if $Y \subseteq \mathcal{U}$ is such that $Y \cap X_\Gamma = \emptyset$, then Y is very new for , . To see this, take any substitution s with domain Y. Then for all $a \in A$, $[s]a = a$. For all $c \in b$ and $a \in A$,

$$\begin{aligned} [s]\langle a, c\rangle &= [s]\{\{a\},\{a,c\}\} \\ &= \{[s]\{a\},[s]\{a,c\}\} \\ &= \{\{a\},\{a,[s]c\}\} \\ &= \langle a,[s]c\rangle \end{aligned}$$

[1]The terminology of *very new* comes from [VC], where there is also a condition called *new* for an operator. This weaker notion is obtained by dropping condition (2) from the definition of very newness. Theorem 2.7 holds for the smooth operators, but the Corecursion Theorem and the main results of this paper require (2) in the definition of very newness.

It follows that $[s](A \times b) = A \times [s]b$. Further, if $[s]_b$ is injective, so is $[s]_{A \times b}$. This is how we check that the operator $, b = A \times b$ is smooth.

The smooth operators are closed under composition. The identity is not proper, hence not smooth. The map $a \mapsto \{x\}$ is not monotone (but almost all urelements are very new for it). An example of monotone, proper, but non-smooth operator would be $a \mapsto a \rightharpoonup a$, the set of partial functions from a to itself. Another is $a \mapsto \mathcal{P}(a) \cup \mathcal{P}(\mathcal{P}(a))$.

The point of proper maps is to avoid the situation that we saw with the identity map above, where $,^*$ was too big to be a final coalgebra. Properness also is related to the condition on substitutions that we saw in point (2) of the statement of *AFA*: systems like $x = x$ whose right-hand sides contain urelements cannot have unique solutions.

Theorem 2.7 ([VC]) *Let , be smooth, and let $e : X \to , X$ be a substitution where X is very new for , . Then e has a unique solution, say s, and $[s]X \subseteq ,^*$. Conversely, if $a \subseteq ,^*$, then there are X, e, and s as above such that $a \subseteq [s]X$. Therefore,*

$$,^* \quad = \quad \bigcup\{[s]X \mid \text{ for some very new } X \text{ for } , \text{, and some } e : X \to , X,\ s \text{ is the solution to } e\}$$

Proof The map e has a unique solution by *AFA* and properness. We also have

$$\begin{array}{rcll} [s]X & = & [s][e]X & \text{since } s \text{ is a solution of } e \\ & \subseteq & [s](, X) & \text{since } [e]X \subseteq , X \\ & = & ,([s]X) & \text{since } X \text{ is very new for } , \end{array}$$

So by the representation formula (4), $[s]X \subseteq ,^*$.

For the converse assertion, we first find b so that $a \subseteq b \subseteq , b$. (This takes an argument using the Collection Axioms and (4).) By smoothness, let $d : X \to b$ be a bijection from some set $X \subseteq \mathcal{U}$ which is very new for , . Then by smoothness again, the inclusion $b \subseteq , b$ lifts to a map $e : X \to , X$. It is not hard to show that the solution to e is exactly d. So $[s]X = [d]X = b$. $\dashv$

Definition Let C be a set or class, and let , be an operator. A *, -notation scheme for C* is a bijection $den : X \to C$ whose domain X is a set or class of urelements which is very new for , .

Proposition 2.8 *For every set or class C and every smooth operator , , there is a , -notation scheme for C, say $den : X \to C$, such that $[den]X = C$.*

Proof Let $Y \subseteq \mathcal{U}$ be a set with the property that the complement $\mathcal{U} - Y$ is very new for , . Let $f : C \to \mathcal{U}$ be given by $f(c) = \mathsf{new}_Y(c)$. Then f is injective. Let $X = f[C]$, and let den be the inverse of f. $\dashv$

Definition Let , be a smooth operator, let $f : C \to , C$ be a , -coalgebra, and let $den : X \to C$ be a , -notation scheme. A map $\varphi : C \to ,^*$ *satisfies , -corecursion for f relative to den* if for all $c \in C$,

$$\varphi(c) \quad = \quad [\varphi \circ den]_{\Gamma X} \circ [den]^{-1}_{\Gamma X} \circ f \tag{6}$$

Theorem 2.9 (Corecursion Theorem for smooth operators, [VC]) *Let , be a smooth operator, and let $f : C \to , (C)$. There is a unique map*

$$\varphi : C \to ,^*$$

such that φ satisfies , -corecursion for f relative to some , -notation scheme with range C. Furthermore, φ satisfies , -corecursion for f relative to any *, -notation scheme with range C.*

It is argued in [VC] that this principle justifies most of the known instances of corecursion. There are also extensions of the result to corecursion in parameters and to simultaneous corecursion. We will not detail these here, but we will give an example of how the theorem is used.

Example 2.3 Let , be the smooth operator given by $, b = A \times b$. Let $Tr(A)$ be the greatest fixed point of the operator $\Delta b = A \times b \times b$. These sets are finite; indeed when the elements of A are hereditarily finite sets so are the elements of $Tr(A)$. Nevertheless, we think of the elements of $Tr(A)$ as infinite binary trees each of whose nodes are labeled with an element of A. Let $C = Tr(A)$, and consider the , -coalgebra $f : C \to , C$ given by

$$f(t) \quad = \quad \langle 1^{st}t, 2^{nd}t \rangle.$$

Reformulating things a bit, $f(\langle a, t_1, t_2 \rangle) = \langle a, t_1 \rangle$. Then by the Corecursion Theorem we get a map $\varphi : C \to ,^*$ satisfying (6). We simplify this to calculate a convenient formula for φ.

Fix a notation scheme $den : X \to C$. For $t \in C$, we write x_t instead of $den^{-1}(t) = [den]_X^{-1} t$. Then $[den]_{\Gamma X}^{-1} \circ f$ is given by

$$[den]_{\Gamma X}^{-1} \circ f(t) \quad = \quad \langle 1^{st}t, x_{2^{nd}t} \rangle.$$

Note that on the right we have $1^{st}t$, an element of A, and then the element of X corresponding $2^{nd}t \in Tr(A)$. Further $, X = A \times X$, and we need to consider $[\phi \circ den]_{\Gamma X}$. Since X is very new for , , $[\phi \circ den]$ fixes the elements of A. For $a \in A$ and $t \in C$, $[\phi \circ den]\langle a, x_t \rangle = \langle a, \phi(t) \rangle$. (Details from Example 2.2 are used here.) So for all $t \in C$,

$$\begin{array}{rcl} \varphi(t) & = & [\phi \circ den] \left([den]_{\Gamma X}^{-1} \circ f(t)\right) \\ & = & [\phi \circ den] \langle 1^{st}t, x_{2^{nd}t} \rangle \\ & = & \langle 1^{st}t, \varphi(2^{nd}t) \rangle \end{array}$$

Once again, we can also reformulate this to $\varphi(\langle a, t_1, t_2 \rangle) = \langle a, \varphi(t_1 \rangle$. The upshot is that φ takes a tree to its *leftmost branch.*

Note that φ was defined after we had specified the denotation scheme. Of course, we want to be sure that the function is independent of the denotation scheme, and indeed that it is the only function satisfying the equation which we derived for it. These points are all part of the statement of Theorem 2.9. This accounts for the long-windedness in the statement of the theorem, a feature which is to some extent avoided in the category-theoretic formulations.

This example illustrates the utility both of functions defined by corecursion and of our general theory in justifying the definitions of such functions.

2.5 Smoothness and map uniformity

Now that we stated Corecursion Theorems both for functors which are uniform on maps and for smooth operators, we might go back and explore the relation between the two concepts. First, an important technical result.

Proposition 2.10 *Suppose that F and G are substitution-like, and $X \subseteq \mathcal{U}$ is such that for all $x \in X$, $F(x) = G(x)$. Then for all sets a such that support$(a) \subseteq X$, $F(a) = G(a)$.*

Proof We sketch the proof for those familiar with bisimulations.

The hypothesis implies that the class of pairs

$$\{\langle F(a), G(a)\rangle \mid a \in V_{afa}[X]\}$$

is a bisimulation relation on sets. AFA implies that every such class is a subrelation of the identity. This means that for all $a \in V_{afa}[X]$, $F(a) = G(a)$. $\dashv$

Theorem 2.11 *Let , be an operation on sets, C a proper class of urelements. The following are equivalent:*

- *(A) , is monotone and all subsets of C are very new for , .*
- *(B) , is the object part of a (unique) functor with the following property: whenever X is a subset of C and $s : X \to b$, , s : , $X \to$, b is given by , $s = [s]_{\Gamma X}$.*

Before proving this theorem, a couple of remarks are in order. The part of "smoothness" mentioning very new urelements is not quite the assertion (A) of the last theorem. The difference is that (A) requires a proper class of urelements, while smoothness strengthens this to insist on "almost all" urelements. The extra strengthening seems harmless in practice, since we know of no natural examples which have condition (A) but are not smooth. Furthermore, the stronger condition of smoothness allows us to prove that the operators involved are closed under composition. This is quite useful, since we want to deal with operators like $a \mapsto \mathcal{P}(A \times (B \cup a))$, which are best analyzed as the compositions of simpler operators. Another point is that properness is not part of (A) or (B). Indeed the identity satisfies both parts. The monotonicity hypothesis is needed in (A) as the example $b \mapsto \{b\}$ shows. That is, the part of (A) dealing with new urelements does not itself imply monotonicity. Finally, note that if (B) holds for a class C which is the complement of a set in UU, then , is map uniform.

We now give the proof of Theorem 2.11, beginning with a proof of (A) $\Rightarrow$ (B). Given an operator , on sets, we turn , into a functor as follows. Given $f : a \to b$, let $d : X \to a$ be a bijection, where $X \subseteq C$. Then define , f by

$$, f \quad = \quad [f \circ d]_{\Gamma X} \circ [d]^{-1}_{\Gamma X}. \tag{7}$$

Note that we are using the "very new" property here to get an inverse to $[d]_{\Gamma X}$. Before going any further, we need to be sure that (7) gives a map , f : , $a \to$, b. The critical point is the codomain of the map, and this is precisely where the monotonicity of , is used. Let E be the image of $[f \circ d]_{\Gamma X}$; we need only check that $E \subseteq\ ,\ b$. Note that $[f \circ d]X \subseteq b$. So by the fact that X is very new for , (and hence , commutes with $[f \circ d]$) and monotonicity,

$$E \quad = \quad [f \circ d](,\ X) \quad = \quad ,\ ([f \circ d]X) \quad \subseteq \quad ,\ (b).$$

In the case when $a = X$ and d is the identity (so that f corresponds to the substitution s in the statement of (B)), equation (7) reduces to the formula in (B) concerning the action of , on substitutions. In the case that f is the identity map, then (7) shows that $,\ f$ too is the identity. We therefore need only check that , is well-defined (i.e., independent of the choice of X and d) and preserves composition. We first prove a technical result and then return to the remaining verifications.

Claim For all $X \subseteq C$, $\mathit{support}(,\ X) \cap C \subseteq X$.

To prove this, suppose toward a contradiction that $y \in (\mathit{support}(,\ X) \cap C) - X$. Let $a \in\ ,\ X$ be such that $y \in \mathit{support}(a)$. Since $\mathit{support}(,\ X)$ is a set, let $z \notin \mathit{support}(,\ X)$. Let s be the substitution $\{\langle y, z\rangle\}$. The domain is $\{y\} \subseteq C$. Since $y \notin X$, $[s]X = X$. So by the "very new" condition, $,\ X =\ ,\ ([s]X) = [s](,\ X)$. But since $y \in \mathit{support}(,\ X)$, we have $z = s(y) \in \mathit{support}([s](,\ X))$. Thus $z \in \mathit{support}(,\ X)$. This contradicts the choice of z.

This proves the claim, so we return to the verification that the value of (7) is independent of the choice of $d : X \to a$. Suppose that we have $f : a \to b$, and two bijections $d : X \to a$ and $e : Y \to a$, where X and Y are subsets of C. Let $\alpha : Y \to X$ be $d^{-1} \circ e$. We will show that $[d]_{\Gamma X} \circ [\alpha]_{\Gamma Y} = [e]_{\Gamma Y}$ and that $[f \circ d]_{\Gamma X} \circ [\alpha]_{\Gamma Y} = [f \circ e]_{\Gamma Y}$. From these two facts, it follows that

$$\begin{array}{rcl} [f \circ d]_{\Gamma X} \circ [d]_{\Gamma X}^{-1} & = & [f \circ d]_{\Gamma X} \circ [\alpha]_{\Gamma Y} \circ [e]_{\Gamma Y}^{-1} \\ & = & [f \circ e]_{\Gamma Y} \circ [e]_{\Gamma Y}^{-1} \end{array}$$

Note that we use the second part of the "very newness" assumption in asserting that $[e]_{\Gamma Y}$ is invertible. The reason that $[d]_{\Gamma X} \circ [\alpha]_{\Gamma Y} = [e]_{\Gamma Y}$ is as follows. Let $D = (\mathcal{U} - C) \cup Y$. By the claim, $,\ Y \subseteq V_{\mathit{afa}}[D]$. The operations $[d \circ \alpha]$ and $[e]$ are substitution-like, and they agree on Y. They also agree on all urelements outside of C. (Note that they need not agree on X.) Hence they agree on all elements of D. By Proposition 2.10, they agree on all of $V_{\mathit{afa}}[D]$, and in particular on $,\ Y$. Hence

$$\begin{array}{rcl} [e]_{\Gamma Y} & = & \{\langle c, [d][\alpha]c\rangle \mid c \in\ ,\ Y\} \\ & = & \{\langle a, [d]a\rangle \mid a \in\ ,\ X\} \circ [\alpha]_{\Gamma Y} \\ & = & [d]_{\Gamma X} \circ [\alpha]_{\Gamma Y} \end{array}$$

In the calculation above, we also used "very newness" assumption (1) to see that $[\alpha]_{\Gamma Y}$ is surjective. Since $f \circ (d \circ \alpha) = f \circ e$, the same kind of reasoning shows that $[f \circ d]_{\Gamma X} \circ [\alpha]_{\Gamma Y} = [f \circ e]_{\Gamma Y}$. This completes the proof that , is well-defined on morphisms.

The proof that , preserves composition is similar. Suppose that $f : a \to b$ and $g : b \to c$. Let $d : X \to a$ and $e : Y \to b$ be bijections from subsets of C. Then $,\ f = [f \circ d]_{\Gamma X} \circ [d]_{\Gamma X}^{-1}$ and $,\ g = [g \circ e]_{\Gamma Y} \circ [e]_{\Gamma Y}^{-1}$. By our claim, we see that

$$[f \circ d]_{\Gamma X} \quad = \quad [e]_{\Gamma Y} \circ [e^{-1} \circ f \circ d]_{\Gamma X}.$$

(The formal details parallel the last paragraph: The operations $[f \circ d]$ and $[e] \circ [e^{-1} \circ f \circ d]$ agree on D, hence on $V_{\mathit{afa}}[D]$. This class includes $,\ X$.) And similarly, $[g \circ e]_{\Gamma Y} \circ [e^{-1} \circ f \circ d]_{\Gamma X} =$

$[g \circ f \circ d]_{\Gamma X}$. Therefore

$$
\begin{aligned}
, g \circ , f &= [g \circ e]_{\Gamma Y} \circ [e]^{-1}_{\Gamma Y} \circ [f \circ d]_{\Gamma X} \circ [d]^{-1}_{\Gamma X} \\
&= [g \circ e]_{\Gamma Y} \circ [e]^{-1}_{\Gamma Y} \circ [e]_{\Gamma Y} \circ [e^{-1} \circ f \circ d]_{\Gamma X} \circ [d]^{-1}_{\Gamma X} \\
&= [g \circ f \circ d]_{\Gamma X} \circ [d]^{-1}_{\Gamma X} \\
&= , (g \circ f)
\end{aligned}
$$

This shows that as an operator on sets, , extends to a functor satisfying $, s = [s]_{\Gamma dom(s)}$ for all substitutions s defined on subsets of C. In fact, the extension is unique. This is because (7) is a consequence of functoriality. So we have extended , to a functor in the only possible way.

The proof of (B) $\Rightarrow$ (A) also takes a few steps. First, we obtain a general formula for the action of , on arbitrary morphisms. Let $f : A \to B$. We can find a set $X \subseteq C$ and a bijection $d : X \to A$. Then by functoriality we have that

$$, f \circ [d]_{\Gamma X} = , f \circ , d = , (f \circ d) = , [f \circ d]_X = [f \circ d]_{\Gamma X}.$$

Every endofunctor on Set or Class has the property that if f is a bijection, so is $, f$. (This follows from the fact that the bijections are exactly the invertible morphisms in the category. Thus $, d = [d]_{\Gamma X}$ is a bijection. This implies that

$$, f \quad = \quad [f \circ d]_{\Gamma X} \circ [d]^{-1}_{\Gamma X},$$

which is precisely Equation (7). This immediately tells us that , is standard, and hence monotone, as follows. If f happens to be an inclusion, then d and $f \circ d$ agree on all urelements. So the substitution-like operations $[f \circ d]_{\Gamma X}$ and $[d]_{\Gamma X}$ agree on all sets, and therefore, $, f$ will be the inclusion $i(, A, , B)$. This shows that , is standard.

Claim For all $X \subseteq C$, $support(, X) \cap C \subseteq X$.

To prove this claim, assume towards a contradiction that $y \in (support(, X) \cap C) - X)$. Let $a \in , X$ be such that $y \in support(a)$. Since X is a set, let $z \notin X \cup \{y\}$. Let $t : X \to X \cup \{y\}$ be the inclusion, and let $s : X \cup \{y\} \to X \cup \{z\}$ be the identity on X, $s(y) = z$. Then $s \circ t$ is an inclusion, so

$$, s \circ , t \quad = \quad , (s \circ t) \quad = \quad i(, X, , (X \cup \{z\})). \tag{8}$$

Similarly, $, t = i(, X, , (X \cup \{y\}))$. However, $, s = [s]_{\Gamma(X \cup \{y\})}$. By monotonicity, $a \in , (X \cup \{y\})$. Also, $[s]a \neq a$, since $y \in support(a)$ and $s(y) = z \notin support(a)$. It follows that $, s(, t(a)) \neq a$. But this contradicts the inclusion assertion of (8). This completes the proof of this claim.

Claim Let $X \subseteq C$ and let s be a substitution defined on X. Let a be any set. Then $, [s]_a = [s]_{\Gamma a}$.

Before proving this, we note that this claim finishes the proof of (B) $\Rightarrow$ (A) in this theorem: we already have shown that , is monotone, and so we only need to show that C is a class of urelements which is very new for , . If $X \subseteq C$ and a is any set, then $, ([s]a) = [s](, a)$ because these sets are the codomains of the morphisms in the claim. Furthermore, since , preserves bijective maps, we see that if $[s]_a$ is a bijection, so is $[s]_{\Gamma a}$.

Now we prove the claim. Let $d : Y \to a$ be a bijection, where $Y \subseteq C$. By (7),

$$, ([s]_a) \circ [d]_{\Gamma Y} \quad = \quad [[s]_a \circ d]_{\Gamma Y}. \tag{9}$$

Consider the two substitution-like operators $[s] \circ [d]$ and $[[s]_a \circ d]$. These agree on the elements of Y. They also fix all $z \in \mathcal{U} - C$. So they agree on all elements of *support*$(, Y)$ by the first claim. By Proposition 2.10,

$$[[s]_a \circ d]_{\Gamma Y} \quad = \quad [s]_{\Gamma a} \circ [d]_{\Gamma Y}.$$

From this and (9), $[s]_{\Gamma a} \circ [d]_{\Gamma Y} = , ([s]_a) \circ [d]_{\Gamma Y}$. As we know, $[d]_{\Gamma Y}$ is a bijection. Therefore $[s]_{\Gamma a} = , ([s]_a)$, as desired. This finishes the proof of the claim and hence the proof of the theorem.

To get our next result, we first need a general result on endofunctors on Set and Class.

Proposition 2.12 *Let , be an endofunctor on* Set *which is monotone and standard. Then , extends uniquely to an endofunctor Δ on* Class *which is also monotone and standard. Moreover, if , is map uniform, then so is Δ.*

Proof Define $\Delta(C) = \bigcup\{, (a) \mid a \text{ is a subset of } C\}$. It is easy to check that Δ is monotone on classes. For $f : C \to D$, define

$$\Delta f \quad = \quad \bigcup\{, (f \restriction a) \mid a \text{ is a subset of } C\}.$$

To see that this makes sense, we need to know that if $a \subseteq b$, then $, (f \restriction a)$ is a subfunction of $, (f \restriction b)$. To see this, we use the standardness and functoriality of , :

$$, (f \restriction a) \quad = \quad , ((f \restriction b) \circ i(a, b)) \quad = \quad , (f \restriction b) \circ i(, a, , b).$$

The uniqueness assertion concerning Δ is easy, as is the standardness property. The map uniformity comes from the fact that $[s]_C = \bigcup\{[s]_a \mid a \subseteq C\}$. ⊣

Corollary 2.13 *If , is a smooth operator, then , is the object part of an endofunctor on* Class *that is proper and map uniform. Conversely, if , is an endofunctor on* Class *which is proper and map uniform, then as an operator on sets , is smooth.*

Proof Let , be smooth. By Proposition 2.12, we need only extend it to an endofunctor on Set which is map uniform and standard. , satisfies condition (A) of Theorem 2.11 with C as the proper class $\mathcal{U} - X_\Gamma$. So by the theorem, , does extend to an endofunctor which is map uniform. The proof also gives standardness. (It is also easy to verify that map uniformity implies standardness directly.)

The other direction is an immediate consequence of (B) ⇒ (A) of Theorem 2.11. ⊣

3 A unified, near-categorial foundation for corecursion

In this section we want to take seriously the ideological point about smooth operators. We present a category C of "classes together with notation systems."

Objects of C are triples (a, X, π) such that a is a class, $X \subseteq \mathcal{U}$, and the substitution $\pi : X \to a$ is a bijection.

Morphisms of C A morphism from (a, X, π) to (b, Y, σ) is a substitution $f : X \to b$. Usually we just identify the morphism with the substitution, though technically one would want to incorporate the domain and codomain. The same technicality arises with the category of sets, of course.

The identity morphism on the object (a, X, π) is just π.

We'll write the composition operation with the $\cdot$ symbol instead of $\circ$ to avoid confusion. Suppose $f : (a, X, \pi) \to (b, Y, \sigma)$ and $g : (b, Y, \sigma) \to (c, Z, \tau)$. Then define $g \cdot f : (a, X, \pi) \to (c, Z, \tau)$ to be $g \circ \sigma^{-1} \circ f$.

The identity laws are trivial to verify, and the associative law for the composition operation is also straightforward.

Definition A *partial endofunctor* on C is an assignment , of objects and morphisms which preserves "almost all" compositions. More precisely, for each set Z, let C_Z be the full subcategory of C determined by the objects (a, X, s) such that $X \cap Z = \emptyset$. Note that a and X might be proper classes. A partial endofunctor is a then pair $(, , Z)$ such that Z is a set and , is an endofunctor on C_Z.

Note that the partial endofunctors are closed under composition: if $(, , Z)$ and (Δ, Y) are partial endofunctors, then so is $(\Delta \circ , , Y \cup Z)$.

There is a straightforward way to turn each smooth operator , into a partial endofunctor. Take a set Z such that for all $X \subseteq \mathcal{U}$ such that $X \cap Z = \emptyset$, X is very new for , . For $(a, X, \pi) \in \mathsf{C}_Z$, we define , (a, X, π) to be $(, (a), \hat{X}, \hat{\pi})$, where

$$\begin{array}{rcl} \hat{X} & = & \{\mathsf{new}_Z(a') \mid a' \in , (X)\} \\ \hat{\pi}(\mathsf{new}_Z(a')) & = & [\pi]_{\Gamma X} a' \end{array} \tag{10}$$

To define , on morphisms, suppose that $f : (a, X, \pi) \to (b, Y, \sigma)$. Then define , f : , $(a, X, \pi) \to$, (b, Y, σ) by

$$, f \quad = \quad [f]_{\Gamma X} \circ [\pi]^{-1}_{\Gamma X} \circ \hat{\pi}.$$

We check that , preserves identities. If f is the identity morphism on (a, X, π), then $f = \pi$. So as desired , $f = \hat{\pi}$, the identity on , (a, X, π).

We check that , preserves composition. Suppose that $f : (a, X, \pi) \to (b, Y, \sigma)$ and $g : (b, Y, \sigma) \to (c, Z, \tau)$. We need to prove that , $(g \circ f) = (, g) \cdot (, f)$. We calculate

$$\begin{array}{rcl} (, g) \cdot (, f) & = & ([g]_{\Gamma Y} \circ [\sigma]^{-1}_{\Gamma Y} \circ \hat{\sigma}) \circ \hat{\sigma}^{-1} \circ ([f]_{\Gamma X} \circ [\pi]^{-1}_{\Gamma X} \circ \hat{\pi}) \\ & = & [g \circ \sigma^{-1} \circ f]_{\Gamma X} \circ [\pi]^{-1}_{\Gamma X} \circ \hat{\pi} \\ & = & , (g \circ f) \end{array}$$

We have used the fact that $[g]_{\Gamma Y} \circ [\sigma]^{-1}_{\Gamma Y} \circ [f]_{\Gamma X} = [g \circ \sigma^{-1} \circ f]_{\Gamma X}$. This is verified by the same kind of argument that we saw in Theorem 2.11.

3.1 Initial algebras and final coalgebras

By an algebra for a partial endofunctor $(, , Z)$ on C we of course mean an arrow

$$f : (, a, \hat{X}, \hat{\pi}) \to (a, X, \pi)$$

where both objects are taken from C_Z. By a coalgebra for $(,\ ,Z)$ we mean an arrow the other way:

$$f\ : (a, X, \pi) \rightarrow (,\ a, \hat{X}, \hat{\pi}).$$

We need a special algebra, $,_*$ and a special coalgebra $,^*$, corresponding to the least and greatest fixed points of , .

First, let $X = \{\mathsf{new}_Z(a) \mid a \in ,_*\}$, and let π be defined by $\pi(\mathsf{new}_Z(a)) = a$. Then $,_*$ is the algebra

$$,_* \quad \equiv \quad \hat{\pi}\ : (,_*, \hat{X}, \hat{\pi}) \rightarrow (,_*, X, \pi).$$

For $,^*$, we let $X = \{\mathsf{new}_Z(a) \mid a \in ,^*\}$, and let π be defined by the formula above. Then $,^*$ is the coalgebra

$$,^* \quad \equiv \quad \pi\ : (,^*, X, \pi) \rightarrow (,^*, \hat{X}, \hat{\pi}).$$

Morphisms of coalgebras are defined in the usual way. The main result here is that for smooth , , there is an initial algebra and a final coalgebra.

Theorem 3.1 *$,_*$ is an initial algebra.*

This result is essentially Theorem 7.6 of Aczel ([NWF]). The key point is that , is standard.

Theorem 3.2 *Assuming AFA, $,^*$ is a final coalgebra.*

Proof Let $f\ : (b, Y, \sigma) \rightarrow (,\ b, \hat{Y}, \hat{\sigma})$ be a coalgebra. Let $e\ : Y \rightarrow\ ,\ Y$ be $[\sigma]^{-1}_{\Gamma Y} \circ f$. By *AFA*, there is a unique map $s\ : Y \rightarrow\ ,^*$ such that

$$s \quad = \quad [s]_{\Gamma Y} \circ [e]_Y \quad = \quad [s]_{\Gamma Y} \circ [\sigma]^{-1}_{\Gamma Y} \circ f.$$

Now s gives a morphism in C

$$s\ : (b, Y, \sigma) \rightarrow\ ,^* = (,^*, X, \pi),$$

and since , is a functor we also have

$$,\ s\ : (,\ b, \hat{Y}, \hat{\sigma}) \rightarrow (,^*, \hat{X}, \hat{\pi}).$$

Recall that Y is very new for , . So $,\ s$ is given by $[s]_{\Gamma Y} \circ [\sigma]^{-1}_{\Gamma Y} \circ \hat{\sigma}$. We claim that this determines a morphism of coalgebras; i.e., that $,\ s \cdot f = \pi \cdot s$. The reason is that

$$\begin{array}{rcl} ,\ s \cdot f & = & [s]_{\Gamma Y} \circ [\sigma]^{-1}_{\Gamma Y} \circ \hat{\sigma} \circ \hat{\sigma}^{-1} \circ f \\ & = & [s]_{\Gamma Y} \circ [\sigma]^{-1}_{\Gamma Y} \circ f \\ & = & s \\ & = & \pi \circ \pi^{-1} \circ s \\ & = & \pi \cdot s \end{array} \tag{11}$$

These calculations show that for *any* morphism $t\ : (b, Y, \sigma) \rightarrow\ ,^*$,

$$,\ t \cdot f = \pi \cdot t \quad \text{iff as a map in } \mathsf{Set},\ t \text{ is a solution to } e. \tag{12}$$

Together with *AFA*, this proves that s is the unique coalgebra morphism so that $,\ s \cdot f = \pi \cdot s$. $\dashv$

Theorem 3.2 implies the final coalgebra theorems which we have mentioned earlier in the paper.

Proof of Theorem 2.6 Let , be a Class-endofunctor which is map uniform and proper. Let $f : C \to , C$ be a coalgebra for , . Then by Corollary 2.13, as an operator on sets, , is smooth. By Proposition 2.8, let $\sigma : Y \to C$ be a , -notation scheme for C. So σ gives a , -coalgebra in C:

$$f \circ \sigma : (C, Y, \sigma) \to (, C, \hat{Y}, \hat{\sigma})$$

By Theorem 3.2, there is a unique morphism $s : (C, Y, \sigma) \to , ^*$. The morphism condition is that $, s \cdot (f \circ \sigma) = \pi \cdot s$. This means that

$$[s]_{\Gamma Y} \circ [\sigma]^{-1}_{\Gamma Y} \circ f \circ \sigma \quad = \quad s.$$

Using map uniformity and functoriality,

$$, (s \circ \sigma^{-1}) \circ f \quad = \quad [s]_{\Gamma Y} \circ [\sigma]^{-1}_{\Gamma Y} \circ f \quad = \quad s \circ \sigma^{-1}$$

This means that $s \circ \sigma^{-1}$ gives a morphism of coalgebras from f to id_{Γ^*}. The uniqueness assertion is verified similarly.

Proof of Theorem 2.9 Let , be a smooth operator, and let $f : C \to , C$. Let $den : X \to C$ be a , -notation scheme. We first show that there is a map $\varphi : C \to , ^*$ which satisfies , -corecursion for f relative to den. This means that

$$\varphi \quad = \quad [\varphi \circ den]_{\Gamma X} \circ [den]^{-1}_{\Gamma X} \circ f \tag{13}$$

Consider C_Z. We have a morphism of , -coalgebras

$$f \circ den : (C, X, den) \to (, C, \hat{X}, \hat{den})$$

and so by Theorem 2.11 there is a unique morphism $s : (C, X, den) \to , ^*$. As (11) shows, this means that $[s]_{\Gamma Y} \circ [den]^{-1}_{\Gamma Y} \circ f \circ den = s$. Thus $[s \circ den^{-1}]_{\Gamma Y} \circ f = s \circ den^{-1}$. So we let $\varphi = s \circ den^{-1}$ to satisfy (13).

There are a number of uniqueness assertions concerning φ. Suppose that ψ also satisfies (13). Then $\psi \circ den$ would give a morphism of coalgebras:

$$\psi \circ den : (C, X, den) \to , ^*.$$

By finality, $\psi \circ den = \varphi \circ den$. Since den is invertible, $\psi = \varphi$.

To conclude, we must show that this same map φ satisfies (13) relative to any , -notation scheme for C. If $d : Y \to C$ is another, then

$$[\varphi \circ d]_{\Gamma Y} \circ [d]^{-1}_{\Gamma Y} \quad = \quad [\varphi \circ den]_{\Gamma X} \circ [den]^{-1}_{\Gamma X}.$$

This independence assertion is part of Theorem 2.11. So (13) will hold with Y replacing X and d replacing den.

4 On the reduction of corecursion to recursion

Let , be a smooth operator, considered as an endofunctor on Set. We wish to state in a very general way what it would mean to reduce , -corecursion to recursion on some directed complete partial order (dcpo). As we have seen, we can obtain the functions defined by corecursion via final coalgebra theorems. So our general statement builds on that approach.

Reduction Desiderata To reduce , -corecursion to recursion, find

1. A dcpo P and a map $i : ,^{*} \to P$.[2]

2. For each , -coalgebra $f : a \to , a$ there should be a monotone map $H^f : P^a \to P^a$. Here P^a denotes the set of all functions from a to P.

3. Continuing, the set of functions from P^a to itself is itself a dcpo. So H^f has a least fixed point, say H^f_*. By finality, there is a unique morphism of coalgebras $s = s_f : f \to ,^{*}$. Thinking of s_f as a map on Set, we require that $i \circ s_f = H^f_*$ for all f.

Recursion comes into condition (3). The point is that H^f_* is a least fixed point, so it will be calculated by recursion. Thinking of i as a way to view elements of $,^{*}$ as elements of the bigger space P, we get our solution to the coalgebra f not by *AFA* or any other non-standard technique but rather by conventional iteration on the dcpo $P^a \to P^a$.

We show how these requirements are satisfied for the operator $, b = A \times b$. We take P to be the set of finite and infinite sequences from A, modeled as (the wellfounded set of) functions from initial segments of the natural numbers into A. As a set of functions, P is a dcpo when ordered by extension. There is a natural map $i : ,^{*} \to P$ which takes every stream over A to the associated infinite sequence from A.

Consider a , -coalgebra $f : b \to A \times b$. We define $H^f : P^b \to P^b$, using k as a variable over P^b. We specify that for all $c \in b$,

$$(H^f k)(c) \quad = \quad \langle 1^{st} f c \rangle \frown k(2^{nd} f c).$$

The operation on the right is the natural one of prepending an element A in front of a finite or infinite function from P to get another element of P.

We check that $i \circ s = H^f_*$. The ideas will be clearer if we work out an example, since the general case involves more notation that might obscure the ideas. Let A contain three elements a_1, a_2, and a_3. Let $b = \{c, d, e\}$, and let f be given by $f(c) = \langle a_1, d \rangle$, $f(d) = \langle a_2, e \rangle$, $f(e) = \langle a_3, d \rangle$. Then s works as follows:

$$\begin{array}{rcl} s(c) & = & \langle a_1, \langle a_2, \langle a_3, \langle a_2, \langle a_3, \ldots \rangle \rangle \rangle \rangle \rangle \\ s(d) & = & \langle a_2, \langle a_3, \langle a_2, \langle a_3, \ldots \rangle \rangle \rangle \rangle \\ s(e) & = & \langle a_3, \langle a_2, \langle a_3, \langle a_2, \ldots \rangle \rangle \rangle \rangle \end{array}$$

The sets on the right are elements of the nonwellfounded set $,^{*} = A \times ,^{*}$. Now $i \circ s$ just takes c, d, and e to the functions on the natural numbers associated to these streams. We calculate H^f_* by iteration:

$$H^f_* \quad = \quad \lim_{n \to \infty} (H^f)^n (\bot_A),$$

[2] In condition (1), we might also require that for each $a \in ,^{*}$, $i(a)$ is a *maximal element* of P. This natural requirement is satisfied in the construction below. Also, we might permit P to be a proper-class sized dcpo. In that case, we would also need to know that each map H^f has a fixed point. This is not immediate, and indeed there are proper-class dcpo's with monotone maps without fixed points. The easiest example is the ordinals and the map $f(\alpha) = \alpha + 1$.

where $\perp_A$ is the function which takes each of c, d, and e to the empty sequence $\langle\,\rangle$. The first few steps are:

$$\begin{array}{lcccc} (H^f)^0(\perp_A) & = & c \mapsto \langle\,\rangle, & d \mapsto \langle\,\rangle, & e \mapsto \langle\,\rangle \\ (H^f)^1(\perp_A) & = & c \mapsto \langle a_1\rangle, & d \mapsto \langle a_2\rangle, & e \mapsto \langle a_3\rangle \\ (H^f)^2(\perp_A) & = & c \mapsto \langle a_1, a_2\rangle, & d \mapsto \langle a_2, a_3\rangle, & e \mapsto \langle a_3, a_2\rangle \\ \vdots & & & \vdots & \end{array}$$

As we mentioned above, H^f_* is just the limit of the finite approximations. And this is exactly $i \circ s$.

Concerning other smooth operators, it is not hard to extend the method here from streams to trees. It is harder to get an approximation result for operators like , $a = \mathcal{P}(A \cup a)$ or even , $a = \mathcal{P}_{<\omega}(A \cup a)$. There are two ways to do this. First, [MMO] build a dcpo for approximating the hereditarily finite non-wellfounded sets; these are the elements of the greatest fixed point of $\mathcal{P}_{<\omega}(a)$. The method extends to $\mathcal{P}_{<\omega}(A \cup a)$, but it is not known how to extend it further to operators like $\mathcal{P}(A \cup a)$. Their method is based on ordered algebras. In contrast, [VC] has a chapter on modal logic, and one can derive an approximation method from that work. In a nutshell, one takes P to be the complete poset of all sentences of infinitary modal logic with A as the set of atomic propositions, ordered by (semantic) implication. P is a proper class. Although the definition of the map $f \mapsto H^f$ is not much different than in the case of streams, it is much harder to show that H^f has a fixed point; the problem is that P is a proper class, and so the usual fixed point theorems to not apply. However, at the end of the day all parts of the desiderata above are fulfilled.

We leave it as an open question whether it is possible to interpret corecursion by recursion for every smooth operator , . This is the most interesting open problem concerning the foundations of corecursion.

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