Topological Reasoning and The Logic of Knowledge

Andrew Dabrowski* Lawrence S. Moss† Rohit Parikh‡

Abstract

We present a bimodal logic suitable for formalizing reasoning about points and sets, and also states of the world and views about them. The most natural interpretation of the logic is in *subset spaces*, and we obtain complete axiomatizations for the sentences which hold in these interpretations. In addition, we axiomatize the validities of the smaller class of topological spaces in a system we call *topologic*. We also prove decidability for these two systems.

Our results on topologic relate early work of McKinsey on topological interpretations of $S4$ with recent work of Georgatos on topologic.

Some of the results of this paper were presented at the 1992 conference on Theoretical Aspects of Reasoning about Knowledge.

*Department of Mathematics, Indiana University, Bloomington IN 47405 USA

†Department of Mathematics, Indiana University, Bloomington IN 47405 USA

‡Departments of Computer Science, Mathematics and Philosophy, CUNY Graduate Center, New York, NY 10036 USA, and Department of Computer Science, Brooklyn College, Brooklyn, NY 11210 USA.

Contents

1 Introduction

What are fields of mathematics, such as probability theory, point-set topology, and combinatorics, *about*? When asked this, a mathematician is likely to answer that the field is about various mathematical concepts, or about the consequences of some axioms or other. Although this answer would be adequate for many purposes, it misses a deeper answer that areas of mathematics can be seen as repositories for our intuitions about several aspects of ordinary life. For example, combinatorics can be seen as just the mathematical home for intuitions about activities like *counting* and *arranging*. General topology can be seen as the home for intuitions about *closeness*.

The point of this paper is to suggest that simple aspects of topological reasoning are also connected with special-purpose logics of *knowledge*. Our goal is to exhibit a formal system which can express simple reasoning about points and sets of the kind that one finds in the very early parts of general topology. The formal system is a modal logic of the kind found in much recent work on the logic of knowledge. However, our project differs from other work both in the technical details of the logic and in the overall motivation.

There are three sources of motivation for our project. First, we are interested in the project of accounting for mathematical practice in weak logical systems whose primitives are appropriately chosen. Our logical system is intended to be a first step in the direction of accounting for topological reasoning. As it stands, it is far too weak. But we feel that our completeness and decidability results suggest that this system, or something like it, may be on the right track.

To make this point in more detail, we believe that although mathematical reasoning *may* be represented in a relatively strong system like first-order logic, this representation is not always appropriate because first-order logic gives one the ability to make complicated assertions that were not seen to be relevant at the outset; that is, *because* it is universally applicable, its particular application in any setting may be unreasonably expressive. (When we move to higher-order logic, the situation is even worse.) A less expressive system is likely to have a lower complexity. This would correspond to our intuition that certain kinds of reasoning in point-set topology are in some sense *easy*. Our goal is precisely to account for the easy parts of topological reasoning.

Second, our study is a contribution to the ongoing development of logics of knowledge because it has an alternative interpretation in terms of the notion of *effort*. Traditionally ([HM], [Pa], [PR], [CM]), knowledge is defined in terms of the notion of *view*. An individual has a view of the world or state, and what that individual knows is whatever is true in all states which are compatible with its view. Thus, for instance, in distributed computing, the role of view is played by what a processor sees, i.e. its local history. Other notions that arise, like common knowledge, can then be defined in a natural way. Similarly, in Mathematical Economics, the notion of view is formalized through some *partition*. If the actual state is s, then the individual knows only that it is in the equivalence class of s, and this class can then be identified with its view.

However, a notion of *effort* enters in topology. Thus if we are at some point p and make a measurement, we will then discover that we are in some neighborhood u of p, but not know exactly where. If we make my measurement finer, then u will shrink, say, to a smaller neighborhood v. A similar consideration arises also in computation. If we are willing to compute for 10 steps, then we may discover that $f(0) = 1$ and that $f(1) = 2$. However, the computation of $f(2)$ takes more than 10 steps and so we may not know whether we are computing the successor function or not. If, however, we are willing to invest 20 steps, then perhaps we will find that $f(2) = 7$ and so f is not the successor function after all. Note, however, that we *never* can discover that f is the successor function.

Such a situation may also arise in ordinary life. Thus if a policeman is measuring the speed of passing cars, his knowledge is confined to the cars that are in his view, i.e. in this case, those cars that he can see. However, his knowledge of the speeds of the cars that he *can* see will depend on the accuracy of his measuring instrument. He can increase his knowledge *without* changing his view, by just using a more accurate measuring instrument. Nonetheless, his knowledge will generally be such that he can always improve it. This fact forces us to represent the situation using *two* modalities, one for knowledge, which is the usual K, and the other which depends on the effort, and this second modality will denoted by our familiar symbol $\Box$.

The point of these analogies is that often one wants to study a situation where the state of the world is only approximately known, and where partial knowledge of the state of the world is represented in some notion of view, and where discussion of both views and the world is reified in some logical setting. The kinds of logics we study are intended to be vehicles for the study of a very wide range of these kinds of situations. We believe that the intuitions captured in our logic could be used, for example, in getting a formal account of the well-known analogies between recursion theory and descriptive set theory.

A final motivation for our work is the development of logical tools for visual reasoning, and with educational software that goes with this. Shin [Sh] gave a formal account of the use of Venn Diagrams as a tool in elementary reasoning. In a sense, her work is part of an ongoing rehabilitation of diagrammatic reasoning vis a vis purely symbolic manipulation. We believe that elementary topology should be amenable to a similar treatment, since in fact diagrams are as essential in point-set topology as they are in set theory. It is likely that the kind of logical tools we are developing will be useful in an analysis of how diagrammatic reasoning is useful in learning elementary topology.

Comparison with other work There are several places where our work is connected to other logical studies. The project of relating topology to modal logic was begun by McKinsey [McK]. Our language and semantics are more expressive than $S4$ and McKinsey's topological interpretation of it, and in fact we obtain an embedding of the theorems of $S4$ into those of topologic.

Vickers' book [Vi] on the logical and conceptual foundations of topology is a step closer to the kind of notions that we make explicit. In a sense, one could say that our logics allow one to study the aspects of the Stone duality where *shrinking an open* corresponds to gaining information.

Finally, Georgatos [G1, G2] has obtained the completeness theorem for topologic independently. Also the finite model property for topologic was first shown in [G1]. Georgatos has gone on [G3] to study the logic on tree spaces, an interesting interpretation which we have not considered.

Contents of this paper. The logic and its semantics are put forth in Section 1. That section also discusses the main classes of interpretations which we study, and also our axioms concerning them. Finally, we have an extended discussion of two examples in Section 1.3; these examples are used at several points in the paper to provide counterexamples to various assertions.

The main completeness theorem for the subset space logic of Section 2.2 is a direct construction using the properties of maximal consistent sets. A filtration argument for decidability is presented in 2.3. None of the details from these sections is needed for the remainder of the paper, since the arguments for the completeness and decidability results for topologic are quite different.

We study a logic of intersection spaces in Section 2.4. This logic is intermediate between the basic logic of subset spaces and the system of topologic. Unfortunately, we do not have completeness for this system, but we do have some sound axioms which we know not to be complete. We might remark in passing that in some of the possible applications of our work and its eventual extensions, the axiom of intersection will not be so natural. We have in mind settings like quantum mechanics, where the collection of observations would certainly not be closed under intersection.

Finally, we take up topologic in Section 3. We prove completeness and decidability, and we also discuss the relation of topologic to McKinsey's work. All of the results on topologic may be read after Section 2.1.

1.1 A Language and Its Semantics

Definition A *subset frame* is a pair $\mathcal{X} = \langle X, \mathcal{O} \rangle$ where X is a set of *points* and $\mathcal{O}$ is a set of non-empty subsets of X called *opens*. (In contexts farther removed from topology, it might be more suggestive to call these *results* or *evidence*.) It will be convenient to assume that $\emptyset, X \in \mathcal{O}$. $\mathcal{X}$ is an *intersection frame* if whenever $u, v \in \mathcal{O}$ and $u \cap v \neq \emptyset$, then also $v \cap v \in \mathcal{O}$. A *lattice frame* is closed under finite unions, and a *complete lattice frame* is closed under the infinitary intersection and union operations.

We now set up a formal language which is expressive enough for simple arguments concerning subset spaces. Later we shall expand this language.

Definitions Let $\mathcal{A}$ be an arbitrary set of *atomic sentences*. $\mathcal{L}$ is the smallest set containing each $A \in \mathcal{A}$, and closed under the following formation rules: if $\phi, \psi \in \mathcal{L}$, then so are $\phi \wedge \psi$ and $\neg\phi$; if $\phi \in \mathcal{L}$, then $K\phi \in \mathcal{L}$ and $\Box\phi \in \mathcal{L}$.

A *subset space* is a triple $\mathcal{X} = \langle X, \mathcal{O}, \alpha \rangle$, where $\langle X, \mathcal{O} \rangle$ is a subset frame, and $\alpha : \mathcal{A} \to \mathcal{P}X$. If $\langle X, \mathcal{O} \rangle$ is an intersection frame, then $\mathcal{X}$ is called an *intersection space*, and similarly for lattice spaces, etc. (Often we simply speak of *models*.) For $p \in X$ and $p \in u \in \mathcal{O}$, we define the *satisfaction relation* $\models_{\mathcal{X}}$ on $(X \times \mathcal{O}) \times \mathcal{L}$ by recursion on ϕ.

$p, u \models_{\mathcal{X}} A$	iff	$p \in \alpha(A)$
$p, u \models_{\mathcal{X}} \phi \wedge \psi$	iff	$p, u \models_{\mathcal{X}} \phi$ and $p, u \models_{\mathcal{X}} \psi$
$p, u \models_{\mathcal{X}} \neg\phi$	iff	$p, u \not\models_{\mathcal{X}} \phi$
$p, u \models_{\mathcal{X}} K\phi$	iff	$q, u \models_{\mathcal{X}} \phi$ for all $q \in u$
$p, u \models_{\mathcal{X}} \Box\phi$	iff	$p, v \models_{\mathcal{X}} \phi$ for all $v \in \mathcal{O}$ such that $p \in v \subseteq u$

In other words, we are considering a Kripke structure whose worlds are the pairs (p, u) and with two accessibility relations corresponding to shrinking an open ($\Box$) while maintaining a reference point, or to moving a reference point inside the given open (K).

We adopt standard abbreviations: $\phi \vee \psi$ means $\neg(\neg\phi \wedge \neg\psi)$, $L\phi$ means $\neg K \neg\phi$, and $\Diamond\phi$ means $\neg\Box\neg\phi$. So $p, u \models_{\mathcal{X}} L\phi$ if there exists some $q \in u$ such that $q, u \models_{\mathcal{X}} \phi$, and $p, u \models_{\mathcal{X}} \Diamond\phi$ if there exists $v \in \mathcal{O}$ such that $v \subseteq u$ and $p, v \models_{\mathcal{X}} \phi$.

As usual, we write $p, u \models \phi$ if $\mathcal{X}$ is clear from context. We write $p, u \models T$ iff for all $\phi \in T$, $p, u \models \phi$. If $T \subseteq \mathcal{L}$, we write $T \models \phi$ if for all models $\mathcal{X}$, all $p \in X$, and all $u \in \mathcal{O}$, if $p, u \models_{\mathcal{X}} T$, then also $p, u \models_{\mathcal{X}} \phi$. Finally, we also write, $T \models_{Int} \phi$ and $T \models_{top} \phi$ for the natural restriction of this notion to the class of models which are intersection and topological spaces.

Certain kinds of sentences will have special interest for us. Given a model $\mathcal{X}$, and a sentence ϕ, ϕ is *persistent in* $\mathcal{X}$ if for all p, u, v, $p, u \models_{\mathcal{X}} \phi$ implies $p, v \models_{\mathcal{X}} \phi$. (We stress that we only use the notation $p, v \models \phi$ when p belongs to v.) Further, ϕ is *bi-persistent in* $\mathcal{X}$ if for all p, u, v, we have $p, u \models_{\mathcal{X}} \phi$ iff $p, v \models_{\mathcal{X}} \phi$. (We stress that we only use the notation $p, v \models \phi$ when p belongs to v.) It is *(bi)-persistent* if it is (bi-)persistent in all $\mathcal{X}$.

A sentence ϕ is *reliable* in a model $\mathcal{X}$ if $K\phi \to \Box K\phi$ is valid in $\mathcal{M}$ and *reliably known* if it is valid in every $\mathcal{X}$. A sentence of the form $K\Box\phi$ is itself always reliably known. Reliably known sentences represent reliable knowledge and have a rather intuitionistic flavor. However, our logic is classical, since we are trying to represent certain knowledge theoretic ideas in a classical setting, rather than use an intuitionistic setting where such ideas would be *presupposed*. If the topology is discrete, then the only reliable sentences will be persistent. By contrast, with the trivial topology, only *tautologies* will tend to be reliable. Thus, for example, assuming that all boolean combinations of $i(A)$ and $i(B)$ are non-empty, then the only sentences involving A and B which are reliable will be tautologies. Note that when v is a subset of u, then every reliably known sentence satisfied by p, u is also satisfied by p, v confirming our intuition that refining from u to v increases knowledge.

If $\mathcal{X}$ is indeed a topology, then a set $i(A)$ will be open iff every point in $i(A)$ has an open neighborhood contained entirely in $i(A)$ iff at every p in $i(A)$, the sentence $\Diamond KA$ holds. Thus $i(A)$ is *open* iff the sentence $A \rightarrow \Diamond KA$ is valid in the model. We likewise say that ϕ is *open* if $\phi \rightarrow \Diamond K\Box\phi$ is valid. Dually, $i(A)$ is *closed* iff the sentence $\Box LA \rightarrow A$ is valid in the model. It is not hard to see that with the obvious definitions, r.e. subsets of the natural numbers will satisfy the same knowledge theoretic sentence that opens do in a topological setting, and this, we believe, is the source of the similarity. The set $i(A)$ is *dense* iff the sentence LA is valid and it is *nowhere dense* if $\Diamond L\neg A$ is valid.

1.2 The Axioms

The main technical goal of this paper is to axiomatize the validities for several classes of models. We consider the classes of all subset spaces, intersection spaces, lattice spaces, and complete lattice spaces. Here are the basic axioms of our logic:

All substitution instances of tautologies of classical propositional logic

$(A \rightarrow \Box A) \wedge (\neg A \rightarrow \Box\neg A)$ for atomic A

$K\phi \rightarrow (\phi \wedge KK\phi)$

$\Box\phi \rightarrow (\phi \wedge \Box\Box\phi)$

$K(\phi \rightarrow \psi) \rightarrow (K\phi \rightarrow K\psi)$

$\Box(\phi \rightarrow \psi) \rightarrow (\Box\phi \rightarrow \Box\psi)$

$L\phi \rightarrow KL\phi$

$K\Box\phi \rightarrow \Box K\phi$

and we use the following rules of inference

$$\frac{\phi \rightarrow \psi,\ \phi}{\psi}\ MP$$

$$\frac{\phi}{K\phi}\ K\text{-necessitation} \qquad\qquad \frac{\phi}{\Box\phi}\ \Box\text{-necessitation}$$

We call the axioms $K\Box\phi \rightarrow \Box K\phi$, the *Cross Axioms*. The Cross Axioms are perhaps the characteristic axiom of the system, and them are often used the dual form $\Diamond L\phi \rightarrow L\Diamond\phi$. The remaining axioms amount to the assertions that K is $S5$-like, and $\Box$ is $S4$-like.

The axioms and rules of inference are sound for subset spaces. The argument is a routine induction on proofs, and perhaps the only interesting part concerns the Cross Axiom. Let us fix a subset space $\mathcal{X}$ and assume that $p, u \models K\Box\phi$; we claim that $p, u \models \Box K\phi$. To see this, let v be an arbitrary subset of u containing p. We need to see that $p, v \models K\phi$, so let $q \in v$. Then $q \in u$ also, so $q, u \models \Box\phi$. Therefore $q, v \models \phi$. Since q was arbitrary in u, we have $p, v \models K\phi$.

The soundness of the logical system is now a straightforward induction on proofs.

We show in Section 2 that this axiomatization is complete. Following standard methods of modal logic, our proof uses facts concerning the collection of (maximal consistent) theories in the logic. However, a problem that we face is that there does not seem to be any convenient way to define a set-space structure on the m-theories in order to get completeness in a very direct way.

We next consider what happens when $\mathcal{O}$ is required to be closed under intersections; the space is then an intersection space. Here is the simplest new axiom scheme:

Weak Directedness Axioms $\Diamond\Box\phi \rightarrow \Box\Diamond\phi$.

This scheme is obviously sound for intersection spaces, since if $x, u \models \Box\phi$ and v is any other neighborhood of x, then $u \cap v \subseteq u$ and so $x, u \cap v \models \phi$.

As it happens, the Weak Directedness Axioms do not lead to a complete axiomatization of the valid sentences on intersection spaces. We discuss the incompleteness further in Section 1.3, and in Section 2.4 we present some other, more complicated axioms. It is still open to get a complete axiom system for intersection spaces.

Finally, for lattice spaces, we have the following scheme:

Union Axioms $\Diamond\phi \wedge L\Diamond\psi \;\rightarrow\; \Diamond[\Diamond\phi \wedge L\Diamond\psi \wedge K\Diamond L(\phi \vee \psi)]$.

To check soundness on spaces closed under unions, suppose that U, $V \subseteq X$ with $x \in U$ and $y \in V$, and suppose that $x, U \models \phi$ and $y, V \models \psi$. Then there is a neighborhood $W = U \cup V$ of x which contains one open set, U, where ϕ is satisfied and another, V, where ψ is satisfied. Since each point of $U \cup V$ is either in U or V, every point in W has a neighborhood in which either $L\phi$ or $L\psi$ is satisfied. Note that here we require that both U and W contain x.

The system whose axioms are the subset space axioms together with the Weak Directedness Axioms and Union Axiom will be called *topologic*. The idea is that topologic should be strong enough to support elementary topological reasoning. We show in Section 3.3 that the topologic axioms give a complete axiomatization of the validities on topologies and indeed on all lattice spaces. The completeness result goes by considering a canonical model, and it also shows the relation between our work and the older studies of McKinsey and Tarski on modal logic and topology. Indeed, the canonical model turns out to be a complete lattice, and thus our axioms are complete for the classes of topological spaces and complete lattice spaces as well. Topologic has the finite model property (3.5) and is therefore decidable. The valid sentences on subset spaces and intersection spaces do not have this property. However, in Section 2.3 we adapt filtration to show that the subset space validities are decidable.

1.3 Examples

In this section, we present three examples. The first should help the reader to become familiar with the semantics and with the various axioms we study. These other two examples are somewhat pathological, and hence they also help to motivate some of the technicalities of our later constructions.

Example A. Consider the case when X is the set R of real numbers, and $\mathcal{O}$ is the standard topology on R. Suppose that there are two atomic predicates P and I, and that $\alpha(P) = [0,2]$, and $\alpha[I] = \{x \in R : x \text{ is irrational}\}$. Then $(1,(0,3)) \models P$, Also, since $2.5 \in (0,3)$, $(1,(0,3)) \models L\neg P$. Moreover, $(1,(0,3)) \models \Diamond KP$, since we can shrink $(0,3)$ around 1 to $(.5, 1.1)$, say, and have the new neighborhood entirely inside the interpretation of P. This shrinking is possible at all points inside $\alpha(P)$, and so $(0,R) \models K(P \rightarrow \Diamond KP)$. Further, $1, R \models K\Box LI$, since every open set containing any real also contains an irrational. These examples show the connection of the semantics to some of the basic ideas of topology.

Next, we present an example which shows a number of things about the m-theories realized in spaces closed under finite intersections.

Example B. This example will show that the logic of subset spaces does not have the finite model property: there are satisfiable sentences whose models must have infinitely many points. (When we insist on satisfiability in a lattice of sets, then the finite model property holds.) The space $\mathcal{X}$ has points

$$a_0, a_1, a_2, \ldots, a_n, \ldots, \qquad b_0, b_1, b_2, \ldots, b_n, \ldots, \qquad \text{and } c\ .$$

There are several families of opens

$$\begin{aligned} u_n &= \{c\} \cup \{a_m : m \geq n\} \cup \{b_m : m \geq n\} \\ v_n &= \{c\} \cup \{a_m : m > n\} \cup \{b_m : m \geq n\} \\ w_n &= \{a_n, b_n\} \\ w'_n &= \{b_n\} \end{aligned}$$

The points and opens of this space are shown in Figure 2, but we have not indicated the singleton sets w'_n. (We want these sets in order to get a space closed under intersections.)

The language has three predicates A, B, and C; we interpret them by $i(A) = \{a_i : i \in N\}$, $i(B) = \{b_i : i \in N\}$, and $i(C) = \{c\}$. Let

$$\textsf{first-and-}B \quad \equiv \quad B \wedge \Box(KB \vee LC).$$

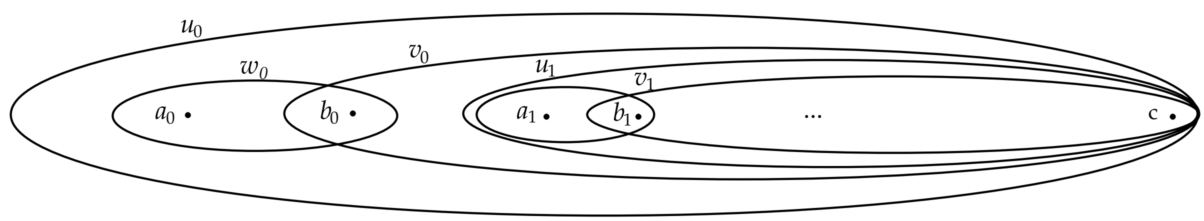


Figure 1: Five Points From Example B

Informally, first-and-B should hold at a pair (d, s) only when d is a b-point, and only when d is the first element of s (the element with lowest subscript). Note that $b_i, v_i \models$ first-and-B, since every subset of v_i which contains b_i is either the singleton $\{b_i\}$ or contains c. Further, $b_i, u_i \models \neg$first-and-B, since $w_i \subseteq v_j$, and $b_i, w_i \models \neg(KB \vee LC)$. Finally, if $i > j$, then $b_i, v_j \models \neg$first-and-B for the same reason. So in this space first-and-B has the meaning that we have described.

Another fact about this example is that if $i > j$ and $i' > j'$, then $th(a_i, u_j) = th(a_{i'}, u_{j'})$, and $th(a_i, v_j) = th(a_{i'}, v_{j'})$. Similar statements hold for the b-points. Also, $th(a_i, u_i) = th(a_{i'}, u_{i'})$, $th(b_i, v_i) = th(b_{i'}, b_{i'})$, $th(c, u_i) = th(c, u_{i'})$, and $th(c, v_i) = th(c, v_{i'})$. All of these facts are proved by induction on the language $\mathcal{L}$. Alternatively, one may use the appropriate version of Fraisse-Ehrenfeucht games for this semantics. A third way is to use bisimulations, and we return to this example in Section 2.3 to verify these facts.

The main interest of this model is that it shows that the set space logic does not have the finite model property. To see this, consider $\phi = L($first-and-$B)$, and also consider the theories $T = th(c, u_i)$ and $U = th(c, v_i)$. Both T and U contain

$$\psi \quad \equiv \quad (\Box\Diamond\phi) \wedge (\Box\Diamond\neg\phi). \tag{1}$$

We claim that no sentence of the form (1) can have a finite model. For suppose that $\mathcal{X}$ were a finite space containing x and u such that $x, u \models \psi$. We may assume that u is a $\subseteq$-minimal open about x with this property. But the minimality implies that $x, u \models \phi \wedge \neg\phi$, and this is absurd.

Example C. This example will show that the subset space axioms together with the Weak Directedness Axioms do not give an axiomatization of the validities of intersection spaces. It also shows that an alternative version of the Union Axioms is properly weaker than that scheme. Let $X = \{a, p, q, z_1, z_2\}$, and let $\mathcal{O} = \{X, u_1, u_2, v_1, v_2\}$, where

$$\begin{array}{ll} u_1 = \{a, p, z_1\} & u_2 = \{a, q, z_2\} \\ v_1 = \{a, z_1\} & v_2 = \{a, z_2\}. \end{array}$$

Let A, P, Q, and Z be atomic sentences; we form a subset space $\langle X, \mathcal{O}, \alpha\rangle$ via

$$\alpha(A) = \{a\} \qquad \alpha(P) = \{p\} \qquad \alpha(Q) = \{q\} \qquad \alpha(Z) = \{z_1, z_2\}.$$

(See Figure 1.)

Our first observation is that the Weak Directedness Axioms are valid in this model (despite the fact that the model is not actually closed under intersections). This is an instance of a more general fact. In a subset frame with only finitely many opens, every open u about a point p can be shrunk to a minimal open v about p. When v is minimal (p, v) automatically satisfies all sentences of the form $\phi \rightarrow \Box\phi$. Moreover, $p, X \models \Diamond\Box\phi$ iff there is some minimal v about p so that $p, v \models \phi$.

Suppose further that for all points p of the space, all minimal opens v containing p are isomorphic. Then if u and v are minimal about p, $th(p, u) = th(p, v)$, where $th(p, u)$ and (p, v) are the sets of sentences which hold at the respective pairs. It follows that if $p, X \models \Diamond\Box\phi$, then $p, X \models \Box\Diamond\phi$. From this it follows that (p, X) validates all Weak Directedness Axioms.

Now in the space under consideration, we do indeed have the minimal neighborhood property. The main point is that the two minimal neighborhoods about a are isomorphic. In this way, for each (p, u) from this space and all the Weak Directedness Axioms ϕ, $p, u \models \phi$.

Consider next the following scheme:

Weak Union Axioms $L\Diamond\phi \wedge L\Diamond\psi \;\rightarrow\; L\Diamond[L\Diamond\phi \wedge L\Diamond\psi \wedge K\Diamond L(\phi \vee \psi)]$.

The Weak Union Axioms also hold in this example. Since there are a number of cases, we only present a few of them. Suppose that $p, u_1 \models \phi$ and $q, u_2 \models \psi$. Then for any $x \in X$ we have $x, X \models L\Diamond\chi$, where

$$\chi \quad\equiv\quad L\Diamond\phi \;\wedge\; L\Diamond\psi \;\wedge\; K\Diamond L(\phi \vee \psi)\;.$$

The most interesting case is where, e.g., $p, u_1 \models \phi$ and $z_2, v_2 \models \psi$. Here, z_1, v_1 satisfies ψ as well, since v_1 and v_2 are isomorphic. Thus for any $y \in u_1$ we have $y, u_1 \models \chi$.

However, there are Union Axioms which fail here; for example

$$\begin{array}{rcl} z_2, X & \models & \Diamond K(\neg Q) \;\wedge\; L\Diamond(LP \wedge K\neg Q) \\ \text{but}\quad z_2, X & \not\models & \Diamond[LP \wedge K\neg Q] \end{array}$$

Note that, prefixed by an L, this sentence would be satisfied. This was the key idea for showing that the Weak Union Axiomd are satisfied.

To summarize: this space satisfies the Weak Union Axioms but not the Union Axioms. Hence the former scheme is properly stronger. There is a second reason for introducing this example, having to do with the m-theories realizable in various spaces.

This example will also show why the subset space logic and Weak Directedness Axioms are incomplete for the class of intersection spaces. First, define $P\,\mathsf{not}\,Q$ and $Q\,\mathsf{not}\,P$ as $LP \wedge \neg LQ$ and $LQ \wedge \neg LP$ respectively; then

$$a, u_1 \models P\,\mathsf{not}\,Q, \qquad a, u_2 \models Q\,\mathsf{not}\,P. \tag{2}$$

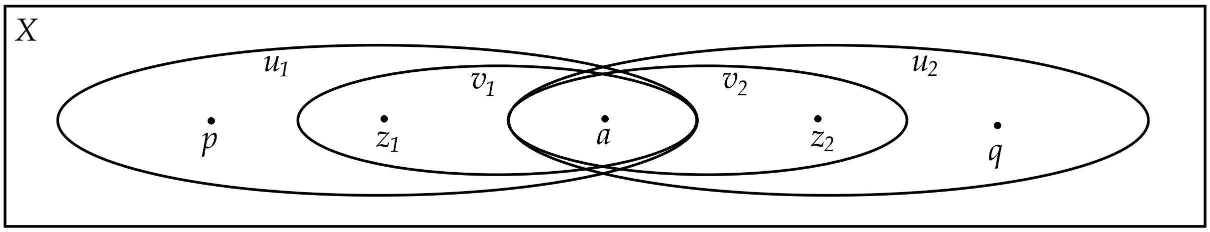


Figure 2: Example C

Therefore

$$a, X \quad \models \quad \Diamond(P \text{ not } Q) \wedge \Diamond(Q \text{ not } P), \tag{3}$$

and since z_1 and z_2 are the only Z-points,

$$a, X \quad \models \quad K\Box LZ \ \wedge \ K(Z \to \neg(\Diamond(P \text{ not } Q) \wedge \Diamond(Q \text{ not } P))) \tag{4}$$

In other words, it is not possible to take a single Z-point z from X and then be able to both shrink X to one open containing a P-point but no Q-points and to take the same z and shrink X to another open containing a Q-point but no P-points. Let ϕ be the conjunction of the sentences in (3) and (4). We claim that ϕ cannot be realized in a space closed under intersection; in fact, the proof shows that it cannot be realized in a space which is downward-directed under inclusion. For suppose that $\mathcal{X}'$ were such a space, and $a', X' \models \phi$. Let u_1' be such that $a', u_1' \models P \text{ not } Q$, and let u_2' be such that $a', u_2' \models Q \text{ not } P$. By hypothesis, we have some $w' \subseteq u_1' \cap u_2'$ which belongs to $\mathcal{O}'$. Since $a', X' \models \phi$, and since $K\Box LZ$ is a conjunct of ϕ, let $z' \in w'$ satisfy Z. Now $w' \subseteq u_1'$, so $z', X' \models \Diamond(P \text{ not } Q)$. Similarly, $z', X' \models \Diamond(Q \text{ not } P)$. But now we contradict (4), since

$$z', X' \quad \models \quad Z \ \wedge \ \Diamond(P \text{ not } Q) \ \wedge \ \Diamond(Q \text{ not } P).$$

From this observation we can see that the subset space axioms together with the Weak Directedness Axioms are not complete for intersection spaces. That is, we have shown that $\models_{int} \neg\phi$. However, ϕ together with all these axioms comprise a consistent set. Since the axioms cannot prove $\neg\phi$, they are not complete.

Concerning unions, note that

$$a, X \ \models \ \Diamond((P \text{ not } Q) \wedge \Diamond K\neg P) \ \wedge \ \Diamond((Q \text{ not } P) \wedge \Diamond K\neg Q) \tag{5}$$

The witnesses are u_1 and u_2. Also

$$a, X \ \models \ K(Z \ \to \ \neg(\Diamond(P \text{ not } Q) \ \wedge \ \Diamond(Q \text{ not } P))) \tag{6}$$

Let ψ be the conjunction of the sentences in (5) and (6). We claim that ψ has no models which are closed under unions. For suppose $\mathcal{X}'$ were such a model. Let u_1' and v_1' be witnesses in $\mathcal{X}'$ to (5); let u_2' and v_2' be subsets witnessing $a, u_1' \models \Diamond K \neg P$ and $a, v_1' \models \Diamond K \neg Q$ respectively. Let $w' = u_1' \cup v_2'$ (we could also use $u_2' \cup v_1'$). Let $z' \in v_2'$ be a Z-point. Then $z', w' \models P \,\mathsf{not}\, Q$ while $z', u_2' \models Q \,\mathsf{not}\, P$. So

$$z', X \ \models\ Z \ \wedge\ \Diamond(P \,\mathsf{not}\, Q) \ \wedge\ \Diamond(Q \,\mathsf{not}\, P)$$

This contradicts (6).

This fact that the model satisfies the Weak Directedness Axioms and Weak Union Axioms shows that the subset space axioms together with all of these axioms cannot refute $\psi \wedge \chi$. Nevertheless $\psi \wedge \chi$ cannot hold in any lattice space (even in any directed space). So the axioms are incomplete. (Nevertheless, we do have completeness for lattice spaces using the stronger Union Axioms.)

2 The Logic of Subset Spaces

In this section, we prove a completeness theorem for the relation $\models$ on the class of subset spaces. We use the subset space axioms alone.

2.1 Properties of Theories

The proof of the Completeness Theorem uses maximal consistent subsets of $\mathcal{L}$, which we call *m-theories.*

Fix a language $\mathcal{L}$, and let $\mathcal{TH}$ be the set of m-theories in $\mathcal{L}$. In the sequel, we use U, V, etc., to denote m-theories. In order to prove that we have given a complete proof system, we need only show that for every m-theory T, there is a subset space model $\mathcal{X} = \langle X, \mathcal{O}, \alpha \rangle$, a point $x \in X$, and a subset $u \in \mathcal{O}$ such that $p, u \models_{\mathcal{X}} T$.

Definitions We define the relation $\xrightarrow{L}$ and $\xrightarrow{\diamond}$ on m-theories by:
$U \xrightarrow{L} V$ iff whenever $\phi \in V$, $L\phi \in U$
$U \xrightarrow{\diamond} V$ iff whenever $\phi \in V$, $\diamond\phi \in U$.
Of course, the maximal consistency of m-theories give other characterizations. For example, $U \xrightarrow{L} V$ if whenever $K\phi \in U$, $\phi \in V$.

Example. The main examples come from the semantics. Let $\mathcal{X} = \langle X, \mathcal{O}, \alpha \rangle$ be a subset space, $u \subseteq v$ belong to $\mathcal{O}$, and let $p, q \in u$. Let $th(p, u)$ be the set of sentences ϕ such that $p, u \models \phi$, and let $th(p, v)$, etc., be defined similarly. Then $th(p, v) \xrightarrow{\diamond} th(p, u)$. (In other words, when we shrink an open about a point go up in $\xrightarrow{\diamond}$.) Also, $th(p, u) \xrightarrow{L} th(q, u)$ and $th(q, u) \xrightarrow{L} th(p, u)$.

Proposition 2.1 *Concerning the relations* $\xrightarrow{L}$ *and* $\xrightarrow{\diamond}$*:*

(1) $\xrightarrow{L}$ *is an equivalence relation.*

(2) $\xrightarrow{\diamond}$ *is reflexive and transitive.*

(3) If $L\phi \in T$*, then there is some* U *so that* $\phi \in U$ *and* $T \xrightarrow{L} U$*.*

(4) If $\diamond\phi \in T$*, then there is some* U *so that* $\phi \in U$ *and* $T \xrightarrow{\diamond} U$*.*

Proof For example, we prove (1). Let U be an m-theory. If $\phi \in U$, then as $\phi \to L\phi$, we also have $L\phi \in U$. This proves that $\xrightarrow{L}$ is relexive. Using the fact that $LL\phi \to L\phi$, we see that $\xrightarrow{L}$ is transitive. Finally, suppose that $U \xrightarrow{L} V$. To show that $V \xrightarrow{L} U$, let $\phi \in U$, If $L\phi \notin V$, then by maximality we must have $K\neg\phi \in V$. $LK\neg\phi \in U$. By S5-ness, $LK\neg\phi \to \neg\phi$. So $\neg\phi \in U$. But this means that U is inconsistent, and this is a contradiction. Hence $\xrightarrow{\diamond}$ is symmetric. In this way, we verify all of the properties of an equivalence relation.

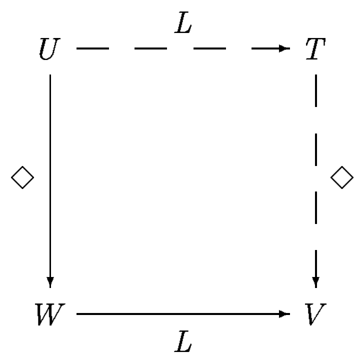
The proof of (2) is similar, using the S4-ness of $\Diamond$. The proofs of (3) and (4) involve arguments using Zorn's Lemma, and a similar proof is given below, in Proposition 2.2. $\dashv$

These facts will be used in the sequel without mention.

Define $U \xrightarrow{L\Diamond} V$ if for all $\phi \in V$, $L\Diamond\phi \in U$. And define relations such as $U \xrightarrow{\Diamond L} V$ and $U \xrightarrow{\Box\Diamond} V$ similarly. For example, $U \xrightarrow{\Diamond L} V$ holds if whenever $\phi \in V$, $\Diamond L\phi \in U$.

The following consequence of the Cross Axioms will be used frequently.

Proposition 2.2 *Let U and V be m-theories, and suppose that there is some W such that $U \xrightarrow{\Diamond} W \xrightarrow{L} V$. Then there is some T so that $U \xrightarrow{L} T \xrightarrow{\Diamond} V$.*



Proof Let $S = \{\Diamond\phi : \phi \in V\} \cup \{\psi : K\psi \in U\}$. We claim that S is consistent; suppose towards a contradiction that it is not. Then there is a finite subset S_0 of S which is inconsistent. Write

$$S_0 \quad = \quad \{\Diamond\phi_1, \Diamond\phi_2, \Diamond \ldots, \Diamond\phi_n\} \cup \{\psi_1, \psi_2, \ldots, \psi_m\},$$

where each ϕ_i belongs to V, and for each j, $K\psi_j \in U$. Let $\phi = \phi_1 \wedge \cdots \wedge \phi_n$, and similarly let $\psi = \psi_1 \wedge \cdots \wedge \psi_m$. V is closed under conjunction, so $\phi \in V$. And since $K\psi$ is equivalent to a conjunction $K(\psi_1 \wedge \cdots \wedge \psi_m)$ from U, we see that $K\psi \in U$. Finally $\Diamond\phi \to \Diamond\phi_i$ for all i, and $\psi \to \psi_j$ for all j. Since S_0 is inconsistent, $\vdash \Diamond\phi \to \neg\psi$. Therefore $\vdash L\Diamond\phi \to L\neg\psi$; hence this sentence belongs to U. And also, $\phi \in V$, so $L\phi \in W$ and $\Diamond L\phi \in U$. So $L\Diamond\phi \in U$ by the Cross Axiom. It follows that $L\neg\psi \in U$. But since $K\psi \in U$, this gives the contradiction that U is inconsistent. So S is consistent. Let $T \supseteq S$ be maximal consistent. By construction, $U \xrightarrow{L} T \xrightarrow{\Diamond} V$. $\dashv$

Proposition 2.2 is the embodiment of the Cross Axiom scheme in the realm of m-theories. The next result is a generalization which will be used in the proof of completeness.

Proposition 2.3 *Suppose that $T_1 \xrightarrow{\Diamond} T_2 \xrightarrow{\Diamond} \cdots T_n$, and suppose that $T_n \xrightarrow{L} U$. Then there are $U_1 \xrightarrow{\Diamond} U_2 \xrightarrow{\Diamond} \cdots U_n$ so that $U_n = U$, and for all i, $T_i \xrightarrow{L} U_i$.*

Proof Given all the m-theories above, we let $U_n = U$. Since $T_{n-1} \xrightarrow{\Diamond} T_n \xrightarrow{L} U_n$, we get U_{n-1} from Proposition 2.2 so that $T_{n-1} \xrightarrow{L} U_{n-1} \xrightarrow{\Diamond} U_n$. We continue backwards in this way, obtaining $U_{n-2}, \ldots, U_1$. $\dashv$

2.2 Completeness of the Subset Space Logic

The goal of this section is to prove the following result:

Theorem 2.4 *The axioms are complete for interpretation in subset spaces. That is, if $T \models \phi$, then $T \vdash \phi$.*

Theorem 2.4 implies the compactness theorem for the subset space logic. Its proof will be given in the course of this section.

The first idea in proving completeness is to consider the canonical model of the logic. This is the set $\mathcal{TH}$ of m-theories together with the relations $\xrightarrow{\Diamond}$ and $\xrightarrow{L}$ defined in Section 2.1. What we would like most is a way to define a family $\mathcal{O}$ of subsets of $\mathcal{TH}$ in order to obtain a subset space. Then, following standard modal completeness proofs, we would hope to show that every m-theory T is the theory of some pair p, u from that model. (Indeed, we might hope that p would be T itself.)

Unfortunately, this idea does not seem to work. The problem is that we do not know any way to define a subset space structure on $\mathcal{TH}$ which leads to completeness. For this reason, we do not approach completeness via the canonical model.

Our strategy is to build a space X of "abstract" points. We shall also have opens given in an abstract way, via a poset P and an antitone (i.e., order-reversing) map $i : \mathsf{P} \to \mathcal{P}^*(X)$. (Here, $\mathcal{P}^*(X)$ is the set of non-empty subsets of X.) The points are abstract since they are not theories. But with each x and each p so that $x \in i(p)$ we shall have a "target" m-theory $t(x, p)$. The goal of the construction is to arrange that in the overall model, $th(x, i(p)) = t(x, p)$.

As usual, in order to prove completeness we need only prove that every m-theory T has a model. Fix such a m-theory T. We build

(1) A set X containing a designated element x_0.

(2) A poset $\langle \mathsf{P}, \leq \rangle$ with least element $\perp$.

(3) A function $i : \mathsf{P} \to \mathcal{P}^*(X)$ such that $p \leq q$ iff $i(p) \supseteq i(q)$, and $i(\perp) = X$. (That is, a homomorphism from $\langle \mathsf{P}, \leq, \perp \rangle$ to $\langle \mathcal{P}^*(X), \supseteq, X \rangle$.)

(4) A partial function $t : X \times \mathsf{P} \to \mathcal{TH}$ with the property that $t(x, p)$ is defined iff $x \in i(p)$. Furthermore, we ensure the following properties for all $p \in \mathsf{P}$, $x \in i(p)$, and ϕ:

(a.1) If $y \in i(p)$, then $t(x,p) \xrightarrow{L} t(y,p)$.

(a.2) If $L\phi \in t(x,p)$, then for some $y \in i(p)$, $\phi \in t(y,p)$.

(b.1) If $q \geq p$, then $t(x,p) \xrightarrow{\diamond} t(x,q)$.

(b.2) If $\diamond\phi \in t(x,p)$, then for some $q \geq p$, $\phi \in t(x,q)$.

(c) $t(x_0, \bot) = T$, where T is the m-theory from above which we aim to model.

Suppose we have X, P, i, and t with these properties. Then we consider the subset space

$$\mathcal{X} \quad = \quad \langle X, \{i(p) : p \in \mathsf{P}\}, \alpha\rangle.$$

where $\alpha(P) = \{x : A \in t(x, \bot)\}$.

Lemma 2.5 (The Truth Lemma) *Assume conditions (1) – (4) for X, P, i, and t. Then for all $x \in X$ and all $p \in \mathsf{P}$ such that $x \in i(p)$,*

$$th_{\mathcal{X}}(x, i(p)) \quad = \quad t(x,p) \ .$$

Proof By induction on ϕ. The atomic case holds by the definition of $\mathcal{X}$, and the induction steps for the boolean connectives are consequences of the fact that the sets in $\mathcal{TH}$ are m-theories.

Suppose that $x, i(p) \models L\phi$. Then there is some $y \in i(p)$ such that $y, i(p) \models \phi$. By induction hypothesis, $\phi \in t(y,p)$. By property (4a.1), $t(x,p) \xrightarrow{L} t(y,p)$. Therefore $L\phi \in t(x,p)$.

On the other hand, if $L\phi \in t(x,p)$, then by property (4a.2) there is some $y \in i(p)$ such that $\phi \in t(y,p)$. By induction hypothesis, $y, i(p) \models \phi$. Therefore $x, i(p) \models L\phi$. This concludes the induction step for L. We complete the proof with the induction step for $\diamond$.

Suppose that $x, i(p) \models \diamond\phi$. Then there is some $i(q) \subseteq i(p)$ such that $x \in i(q)$, and $x, i(q) \models \phi$. By (3), $q \geq p$, and the induction hypothesis implies that $\phi \in t(x,q)$. Finally, by (4b.1), $t(x,p) \xrightarrow{\diamond} t(x,q)$. Therefore, $\diamond\phi \in t(x,p)$.

Going the other way, suppose that $\diamond\phi \in t(x,p)$. Then by (4b.2), there is some $q \geq p$ such that $\phi \in t(x,q)$. By induction hypothesis, $x, i(q) \models \phi$. But $i(q) \subseteq i(p)$ by (3), so $x, i(p) \models \diamond\phi$. $\dashv$

By the Truth Lemma and property (4c) above, $theory_{\mathcal{X}}(x_0, \bot) = T$. So the m-theory that we started with has a model. In the usual way, this proves the Completeness Theorem.

We build X, P, i, and t by recursion. That is, we build approximations X_n, P_n, i_n, and t_n satisfying certain local and global properties. The idea is that if we have a point x and a set u, with a target $t(x,p)$, we should try to insure that $t(x,p)$ is the theory of x,p in the overall completed model. This means that we will need to add points to the model (to witness the $L\phi$ sentences of $t(x,p)$, and we will also have to add elements to the partial order (for the

$\Diamond\phi$ sentences). Of course, when we add points we must also specify their targets relative to all sets to which they belong. So the overall construction generates new requirements as it proceeds.

Fix two objects x_0 and $\bot$. The local properties that the construction will satisfy are as follows, where our numbering schemes are intended to be parallel to the one above for conditions (1)–(4):

(L1) X_n is a *finite* set containing x_0.

(L2) P_n is a finite poset with $\bot$ as minimum, and with the property that for each $p \in \mathsf{P}_n$, the lower set of p, $\{q \in \mathsf{P}_n : q \leq p\}$, is linearly ordered.

(L3) $i_n : \mathsf{P}_n \to \mathcal{P}^{**}(X_n)$, where $\mathcal{P}^{**}(X_n)$ is the collection of subsets of X_n with *at least two* elements. This map i_n has the property that $p \leq q$ iff $i_n(q) \subseteq i_n(p)$; also $i_n(\bot) = X_n$.

(L4) $t_n : X_n \times \mathsf{P}_n \to \mathcal{TH}$ is a partial function with the property that $t_n(x,p)$ is defined iff $x \in i_n(p)$. Furthermore, we assume the following properties for all $x \in X_n$ and $p \in \mathsf{P}_n$:

(a) If $x, y \in i_n(p)$, then $t_n(x,p) \xrightarrow{L} t_n(y,p)$.

(b) If $x \in i_n(q)$ and $q \geq p$, then $t_n(x,p) \xrightarrow{\Diamond} t_n(x,q)$.

(c) $t_0(x_0, \bot) = T$.

We should make a few remarks on these conditions. In (L2), the requirement that the lower sets of points be linear orders is essential to our construction. By maintaining this property throughout the construction, we can use Proposition Of course, $pr[p \wedge q] = pr[q \wedge p]$. efproposition-linear to add points to the model. The condition in (L3) that each $i_n(p)$ have at least two elements is a technical one. It is not really necessary, but it leads to a slight simplification of the overall construction (in Case 2, near the end of the proof).

The global properties are that for all n:

(G1) $X_n \subseteq X_{n+1}$.

(G2) P_{n+1} is an end extension of P_n. That is, P_n is a suborder of P_{n+1}, and if $p \in \mathsf{P}_{n+1}$, $q \in \mathsf{P}_n$ and $p \leq q$, then $p \in \mathsf{P}_n$.

(G3) For all $p \in \mathsf{P}_{n+1}$, $i_{n+1}(p) \cap X_n = i_n(p)$.

(G4) The restriction of t_{n+1} to $X_n \times \mathsf{P}_n$ is t_n.

Finally, our construction has some overall requirements:

(R4a) If $L\phi \in t_n(x,p)$, then for some $m > n$, there is some $y \in i_m(p)$ such that $\phi \in t_m(y,p)$.

(R4b) If $\Diamond\phi \in t_n(x,p$, then for some $m > n$, there is some $q \geq p$ in P_m such that $\phi \in t_m(x,q)$.

Suppose we build X_n, P_n, i_n, and t_n in accordance with the (L), (G), and (R) requirements. Let $X = \bigcup_n X_n$, and let P be the limit of the posets P_n. Let i be defined by $i(p) = \bigcup_{n>m} i_{n+1}(p)$, where m is the least number such that $p \in \mathsf{P}_m$. Finally, let $t(x,p) = t_n(x,p)$, where n is any number such that $t_n(x,p)$ is defined. (The construction has arranged that $t_n(x,p) = t_{n+1}(x,p)$ whenever the latter is defined.)

Proposition 2.6 *Suppose we build X_n, P_n, i_n, and t_n in accordance with the (L), (G), and (R) requirements. Then X, P, i, and t as defined above satisfy conditions (1) – (4) above.*

Proof Obviously, (1) and (2) hold. To check (3), note first that $x \in i(p)$ iff for some n, $x \in i_n(p)$. This implies that if $p \geq q$, then $i(p) \subseteq i(q)$. On the other hand, if $p \not\geq q$, then let n be such that both p, $q \in X_n$. By (G2), $p \not\geq q$ in X_n, so by (L3), let $x \in i_n(p) - i_n(q)$. By (G3), $i_n(q) = X_n \cap i(q)$, so as $x \in i_n(p) \subseteq X_n$, $x \notin i(q)$. Hence $i(p) \not\subseteq i(q)$. It is also easy to see that $i(\bot) = X$. This completes the verification of (3).

The verifications of (4) are consequences of the overall (R) requirements and conditions (G4) and (L4c). ⊣

Now we turn to the details of the construction. At the outset, we fix a map

$$\nu \quad : \quad \omega \;\rightarrow\; \{L, \Diamond\} \times \omega \times \omega$$

with the property that if $\nu(n) = (x,m,k)$ then $m < n$, and for all (x,m,k) there is some $n > m$ such that $\nu(n) = (x,m,k)$. Note that here $\{L, \Diamond\}$ is just a set of symbols; we could use $\{1,2\}$ here instead but prefer to use a more suggestive notation.

We define by recursion on n a tuple $\langle X_n, \mathsf{P}_n, i_n, t_n, \alpha_n, \beta_n\rangle$. The last two items here are functions whose purpose is to insure all of the (R) requirements are met in an orderly fashion in countably many steps. The domain of each α_n and β_n is ω. We shall require that

α_n is a map from ω onto $\{(x,p,\phi) \in X_n \times \mathsf{P}_n \times \mathcal{L} \,:\, L\phi \in t_n(x,p)\}$

β_n is a map from ω onto $\{(x,p,\phi) \in X_n \times \mathsf{P}_n \times \mathcal{L} \,:\, \Diamond\phi \in t_n(x,p)\}$

(Of course, the sets on the right will be countable, so functions α_n and β_n will certainly exist.)

We start with X_0 as a two-element set $\{x_0, x_1\}$. (Making X_0 into a two-element set shortens our argument a trifle later on.) P_0 is the trivial poset $\{\bot\}$, $i_0(\bot) = X_0$, and $t_0(x_0, \bot) = T$. Note that this satisfies all of the (L) requirements. We also fix functions α_0 and β_0 to satisfy the above equations. (Otherwise, α_0 and β_0 are arbitrary.)

Now suppose we are at stage $n+1$ of the construction. There are two cases, depending on the value of $\nu(n+1)$.

Case 1 $\nu(n+1) = (L, m, k)$ for some m and k. Consider $\alpha_m(k)$, and let this triple be (x, p, ϕ). So we have $L\phi \in t_m(x, p)$. Let y be a new point not in X_n. Let $X_{n+1} = X_n \cup \{y\}$. Let $\mathsf{P}_n = \mathsf{P}_{n+1}$. Let

$$i_{n+1}(q) \quad = \quad \begin{cases} i_n(q) \cup \{y\} & \text{if } q \leq p \\ i_n(q) & \text{otherwise} \end{cases}$$

Before going any further, note that conditions (L1), (L2), (G1), (G2), and (G3) are trivial in this case. Also, $i_{n+1}(\bot) = X_{n+1}$. We check that $q \leq r$ iff $i(r) \subseteq i(q)$ by looking at three cases. If neither $q \leq p$ nor $r \leq p$, then $i_{n+1}(q) = i_n(q)$ and $i_{n+1}(r) = i_n(r)$. So we're done by induction hypothesis. If both q and r are below p, then since $i_{n+1}(q) = i_n(q) \cup \{y\}$, and $i_{n+1}(r) = i_n(r) \cup \{y\}$, we see that $i_{n+1}(r) \subseteq i_{n+1}(q)$ iff $i_n(r) \subseteq i_n(q)$. Finally, suppose that $q \leq p$ but $r \not\leq p$. First, $r \not\leq q$, and to be sure $i_{n+1}(q) \not\subseteq i_{n+1}(r)$. Second, since $y \notin i_{n+1}(r)$ and $i_{n+1}(q) = i_n(q) \cup \{y\}$, we again have that $i_{n+1}(r) \subseteq i_{n+1}(q)$ iff $i_n(r) \subseteq i_n(q)$. This completes the verification of (L3).

To define t_{n+1}, we stipulate that (G4) holds. What is left is to define $t_{n+1}(y, q)$ for $q \leq p$. Here we use (L2) to see that $\{q : q \leq p\}$ is a finite, linearly ordered set. Write this set as $q_1 \leq q_2 \leq \cdots \leq q_N = p$. So

$$t(x, q_1) \xrightarrow{\diamond} t(x, q_2) \xrightarrow{\diamond} \cdots \xrightarrow{\diamond} t(x, q_N).$$

Let U be such that $t(x, q_N) \xrightarrow{L} U$ and $\phi \in U$. By Proposition 2.3, there are m-theories $U_1, \ldots, U_N$ so that $U = U_N$, $t(x, q_i) \xrightarrow{L} U_i$ for all i, and $U_1 \xrightarrow{\diamond} \cdots \xrightarrow{\diamond} U_N$. Let $t(y, q_i) = U_i$ for $1 \leq i \leq N$. This definition of t_{n+1} insures (L4b).

To check that (L4a) holds for $n+1$, suppose that $a, b \in i_{n+1}(q)$. We might as well assume that $a = y$, and hence that $q \leq p$. If $b = y$ also, then we have $t_{n+1}(a, q) \xrightarrow{L} t_{n+1}(b, q)$ by the reflexivity of $\xrightarrow{L}$. So we assume that $b \neq y$; i.e., $b \in i_n(q)$. Now $x \in i_{n+1}(q)$, since $q \leq p$. By the definition of t_{n+1}, and by the assumption that (L4a) holds for n, $t_{n+1}(x, q) \xrightarrow{L} t_{n+1}(b, q)$. By construction, and by (G4), $t_{n+1}(x, q) \xrightarrow{L} t_{n+1}(y, q)$. So by symmetry and transitivity, $t_{n+1}(y, q) \xrightarrow{L} t_{n+1}(b, q)$. This completes the verification of (L4a).

Case 2 $\nu(n+1) = (\diamond, m, k)$ for some m and k. This time we consider $\beta_m(k)$, and let this triple be (x, p, ϕ). So $\diamond\phi \in t_n(x, p)$.

Let $y \notin X_n$, and let $q \notin \mathsf{P}_n$. Let $X_{n+1} = X_n \cup \{y\}$. Let $\mathsf{P}_{n+1} = \mathsf{P}_n \cup \{q\}$ with the partial order extended so that for all $r \in \mathsf{P}_n$, $r < q$ in P_{n+1} iff $r \leq p$. Then the new point q is not below any element of P_n, and its lower set is a chain, and we have (L1), (L2), (G1), and (G2). Let $i_{n+1}(q) = \{x, y\}$, and for $r \in X_n$, let

$$i_{n+1}(r) \quad = \quad \begin{cases} i_n(r) \cup \{y\} & \text{if } r \leq p \\ i_n(r) & \text{otherwise} \end{cases}$$

By (L3) for n, $i_n(p)$ contains a point $x' \neq x$. Since $y \neq x'$, $i_{n+1}(q)$ is a proper subset of $i_{n+1}(p)$. For the same reason, if $r \leq q$ in X_{n+1}, then $i_{n+1}(q) \subset i_{n+1}(r)$. But if $r \not\leq q$, then $r \not\leq p$ and hence $y \notin i_{n+1}(r)$. So in this case, $i_{n+1}(q) \not\subseteq i_{n+1}(r)$. This verifies most of (L3), and the remainder of the verification is as in Case 1. Note also that (G3) holds.

It remains to define t_{n+1}, and we start out by stipulating that (G4) holds. For all $r \leq p$ in P_n, let $t_{n+1}(r, y) = t_{n+1}(r, x)$ and $t_{n+1}(r, z) = t_{n+1}(r, x)$. In addition, let $t_{n+1}(q, y) = t_{n+1}(q, x) = U$, where U is any m-theory so that $t_n(x, p) \xrightarrow{\diamond} U$ and $\phi \in U$. The fact that both (L4a) and (L4b) hold for n implies it easily for $n + 1$.

This concludes the definition of X_{n+1}, P_{n+1}, i_{n+1}, and t_{n+1} in either Case 1 or Case 2. To complete the construction in both cases, we need only fix enumerations α_{n+1} and β_{n+1} as above.

This completes the construction. At each step, we have all satisfied each of the (L) and (G) requirements. It remains to check the (R) requirements on the overall construction. We check (R4a); (R4b) is similar. Suppose that $L\phi \in t_n(x, p)$. Let k be such that $\alpha_n(k) = (x, p, \phi)$. Let N be such that $\nu(N) = (L, n, k)$. Then at stage N we insure that there is some $y \in X_N$ such that $\phi \in t_n(y, p)$. In this way, the construction has insured that all of the (R) requirements hold.

This completes the proof of Completeness for subset spaces.

2.3 Decidability of the Subset Space Logic

Despite the failure of the finite model property, it turns out that the logic of subset spaces is decidable. We show this by showing that a satisfiable sentence has a finite *cross axiom* model, defined as follows:

Definition A *cross axiom frame* is a tuple $\langle J, \xrightarrow{L}, \xrightarrow{\diamond} \rangle$ such that J is a set, $\xrightarrow{L}$ is an equivalence relation on J, $\xrightarrow{\diamond}$ is a preorder on J, and the following property holds: If $i \xrightarrow{\diamond} j \xrightarrow{L} k$, then there some l such that $i \xrightarrow{L} l \xrightarrow{\diamond} k$. A *cross axiom model* is a cross axiom frame together with an interpretation α of the atomic symbols of $\mathcal{L}$; α must satisfy the condition that if $i \xrightarrow{\diamond} j$, then $i \in \alpha(A)$ iff $j \in \alpha(A)$.

Note that when we interpret $\mathcal{L}$ on a cross axiom model, we have a single *node* on the left side of the turnstile. That is, we write, e.g., $j \models \phi$ since there are no sets involved.

The subset space logic is sound and complete for interpretations in cross axiom models. Soundness is checked by induction, of course, and we sketch the details concerning completeness. Every subset frame $\mathcal{X} = \langle X, \mathcal{O} \rangle$ gives rise to a cross axiom frame $J_{\mathcal{X}}$, as follows: Let

$$J \quad = \quad \{(p, u) \in X \times \mathcal{O} : p \in u\}.$$

J is just the set of pairs coming from the model; it will serve as the carrier set of the cross axiom frame $J_{\mathcal{X}}$, and to get the rest of the frame, let $(p, u) \xrightarrow{L} (q, v)$ iff $u = v$, and let $(p, u) \xrightarrow{\diamond} (q, v)$ iff $p = q$ and $v \subseteq u$. This defines $J_{\mathcal{X}}$. To turn a subset model $\langle \mathcal{X}, \alpha \rangle$ into a cross axiom model $\langle J_{\mathcal{X}}, \alpha \rangle$, we set $\alpha(A) = \{(p, u) : p \in \alpha(A) \cap u\}$. In other words, $\alpha(A)$

contains the pairs (p,u) such that $p,u \models A$. An easy induction shows that $p,u \models \phi$ (in the subset semantics) iff $p,u \models \phi$ (in the cross axiom semantics). So in this sense, the subset space model and the cross axiom model are equivalent. It follows that if T is satisfiable in a subset space model, then T is satisfiable in a cross axiom model. Therefore, the subset space logic is complete for interpretations in cross axiom models.

Since our language is bimodal and cross axiom models are just (special kinds of) Kripke structures, we have a natural notion of bisimulation. To be precise, if J and K are cross axiom models, then a bisimulation from J to K is a relation $R \subseteq J \times K$ such that

(1) If jRk, then for all atomic symbols A, $j \in \alpha_J(A)$ iff $k \in \alpha_K(A)$

(2) If jRk and $j \xrightarrow{\diamond} j'$, then there is some k' so that $k \xrightarrow{\diamond} k'$ and $j'Rk'$. Conversely, if jRk and $k \xrightarrow{\diamond} k'$, then there is some j' so that $j \xrightarrow{\diamond} j'$ and $j'Rk'$.

(3) The same condition, with $\xrightarrow{L}$ replacing $\xrightarrow{\diamond}$.

Then an easy induction on sentences ϕ shows that if jRk, then $th_J(j) = th_K(k)$. In words, bisimilar points satisfy the same sentences.

Example. This example continues the discussion in Example B by giving the cross axiom model of all m-theories realized in the model. We consider eleven points denoted by $[a,u]$, $[b,u]$, $[c,u]$, $[a,v]$, $[b,v]$, $[c,v]$, $\langle a,u\rangle$, $\langle b,v\rangle$, $\langle a,w\rangle$, $\langle b,w\rangle$, and $\langle b,w'\rangle$. On these points we specify $\xrightarrow{\diamond}$ by

$$\begin{array}{llll}
[a,u] \xrightarrow{\diamond} [a,v] & [a,v] \xrightarrow{\diamond} [a,u] & [a,u] \xrightarrow{\diamond} \langle a,u\rangle & \langle a,u\rangle \xrightarrow{\diamond} \langle a,w\rangle \\
[b,v] \xrightarrow{\diamond} [b,u] & [b,u] \xrightarrow{\diamond} [b,v] & [b,v] \xrightarrow{\diamond} \langle b,v\rangle & \langle b,v\rangle \xrightarrow{\diamond} \langle b,w\rangle \\
[c,u] \xrightarrow{\diamond} [c,u] & [c,v] \xrightarrow{\diamond} [c,u] & \langle b,w\rangle \xrightarrow{\diamond} \langle b,w'\rangle &
\end{array}$$

We also have the identity $\xrightarrow{\diamond}$ arrows, and all the instances of transitivity. For $\xrightarrow{L}$, we relate pairs with the same last letter. E.g., we have $[c,u] \xrightarrow{L} \langle a,u\rangle \xrightarrow{L} [b,u]$.

It is not hard to check that these relations give a cross axiom frame. We interpret the atomic predicates A, B, and C on it in the obvious way: $\alpha(A) = \{[a,u],[a,v],\langle a,u\rangle,\langle a,w\rangle\}$, $\alpha(B) = \{[b,u],[b,v],\langle b,v\rangle,\langle b,w\rangle,\langle b,w'\rangle\}$, and $\alpha(C) = \{[c,u],[c,v]\}$.

We call this cross axiom model $\mathcal{Y}$. Let $\mathcal{X}$ be the model from Example B, considered as a cross axiom model. We obtain a bisimulation between $\mathcal{X}$ and $\mathcal{Y}$ by using all of the following pairs:

$$\begin{array}{llll}
(a_i,u_j)\ R\ [a,u] & (a_i,u_i)\ R\ \langle a,u\rangle & (a_i,v_j)\ R\ [a,v] & (a_i,w_i)\ R\ \langle a,w\rangle \\
(b_i,v_j)\ R\ [b,v] & (b_i,v_i)\ R\ \langle b,v\rangle & (b_i,u_j)\ R\ [b,u] & (b_i,w_i)\ R\ \langle b,w\rangle \\
(b_i,w'_i)\ R\ \langle b,w'\rangle & (c,u_j)\ R\ [c,u] & (c,v_j)\ R\ [c,v] &
\end{array}$$

(We require that $i > j$ so that the data from $\mathcal{X}$ makes sense.) Intuitively, the angular brackets are used when the point is the last element of the set, and the square brackets are used and in the other case. This makes it easy to check that R is a bisimulation.

This bisimulation implies facts about $\mathcal{X}$. For example, $th(a_2, u_2) = th(a_3, u_3)$. It is not hard to check that the theories of all the points in $\mathcal{Y}$ are different; one uses the sentences from Example B.

Finally, $\mathcal{Y}$ is the smallest cross axiom model which we know of with the property that there are j and k with $j \xrightarrow{\diamond} k \xrightarrow{\diamond} j$ with different theories.

Another example of a cross axiom model which is not a subset space is the *canonical model* of the subset space logic:

$$\mathcal{C}(ca) \quad = \quad \langle \mathcal{TH}, \xrightarrow{L}, \xrightarrow{\diamond} \rangle.$$

(The "ca" stands for "cross axiom"; we shall have a number of other canonical models in later sections.) Proposition 2.2 is just the assertion that this tuple actually is a cross axiom model. The standard truth lemma for this structure shows that for all $T \in \mathcal{TH}$, $th(T) = T$; that is, the set of sentences satisfied by the point T in $\mathcal{C}(ca)$ is T itself. We shall develop a theory of filtration on this model to prove a finite model property. However before we turn to that work, we present another example and some general discussion.

We return now to the development of filtration. Let ϕ be any sentence, and consider the following sets:

$\Gamma' = \phi$ together with all of its subsentences.

$\Gamma^{\neg} = \Gamma' \cup \{\neg\phi : \phi \in \Gamma'\}$.

$\Gamma^{\wedge} = \Gamma^{\neg}$ together with all finite conjunctions of distinct elements of $\Gamma^{\neg}$.

$\Gamma = \Gamma^{\wedge} \cup \{L\phi : \phi \in \Gamma^{\wedge}\}$.

Note that all of these sets are finite and closed under subsentences, and also that up to equivalence, $\Gamma^{\neg}$ is closed under negation, $\Gamma^{\wedge}$ under $\wedge$, and Γ under L. It is also convenient to define $\Gamma^{\mathrm{L}} = \Gamma \setminus \Gamma^{\wedge}$. (That is, Γ^{L} is the set-theoretic difference of Γ and $\Gamma^{\wedge}$.)

Of course, all of these classes of sentences depend on the original ϕ, and occasionally we use the notation $\Gamma(\phi)$, etc., to mark this dependence.

Let Δ be any finite set, and let s be a map from a superset of Δ into $\{\mathsf{T}, \mathsf{F}\}$. Define

$$\Delta_s \quad = \quad \bigwedge\{\phi : \phi \in \Delta, s(\phi) = \mathsf{T}\} \;\wedge\; \bigwedge\{\neg\phi : \phi \in \Delta, s(\phi) = \mathsf{F}\}.$$

Note that for all $\phi \in \Delta$, $\Delta_s \vdash \phi$ or $\Delta_s \vdash \neg\phi$ (or both).

This notation Δ_s works for any finite Δ, but we shall only have occasion to use it when Δ is of one of the forms $\Gamma'(\phi)$, $\Gamma^{\neg}(\phi)$, etc.

Here are three useful general facts. First, for all t,

$$\vdash \;\; \Gamma_t \leftrightarrow (\Gamma^{\mathrm{L}}_t \wedge \Gamma^{\wedge}_t). \tag{7}$$

All that we are doing here is separating the conjunction Γ_t into two smaller conjunctions, by collecting all of the elements of Γ_t^{L} in the first conjunct. Second, if Γ_t is consistent then

$$\vdash \ \Gamma_t^{\wedge} \leftrightarrow \Gamma_t'. \tag{8}$$

To see this, note first that $\Gamma_t' \in \Gamma^{\wedge}$. So if Γ_t is consistent, then Γ_t' is a conjunct of $\Gamma_t^{\wedge}$. Also, the consistency implies that no conjunct of $\Gamma_t^{\wedge}$ can contradict Γ_t'; thus all such conjuncts must follow from Γ_t'. Moreover, the fact that $\Gamma_t' \in \Gamma^{\wedge}$ implies that $\Gamma_t^{\wedge}$ can itself be regarded as an element of $\Gamma^{\wedge}$. Our final fact is that

$$\vdash \ \Gamma_t^{\mathrm{L}} \wedge L\Gamma_t^{\wedge} \to L(\Gamma_t^{\mathrm{L}} \wedge \Gamma_t^{\wedge}). \tag{9}$$

The reason for this is that all of the conjuncts of Γ_t^{L} are K or L sentences; so (9) is a consequence of the $S5$ laws.

Definition Γ is *strongly closed under L* if the following condition holds for all maps s and t whose domains include Γ: If $\Gamma_s \wedge L\Gamma_t$ is consistent, then $\vdash \Gamma_s \to L\Gamma_t$.

Proposition 2.7 *For any ϕ, $\Gamma(\phi)$ is a finite set which contains ϕ and which is strongly closed under L.*

Proof Suppose that $\Gamma_s \wedge L\Gamma_t$ is consistent. We claim that s and t agree on Γ^L. For if not, let $L\psi$ be a sentence in Γ^{L} on which s and t disagree. Suppose that $s(L\psi) = \mathsf{T}$ while $t(L\psi) = \mathsf{F}$. Then $\Gamma_s \vdash L\psi$, and $\Gamma_t \vdash K\neg\psi$. By the $S5$ laws, $L\Gamma_t \vdash K\neg\psi$. This implies that $L\Gamma_t$ is inconsistent with Γ_s, and this is a contradiction.

Assume Γ_s. By the version of (7) for s, we have Γ_s^{L}, and since s and t agree on Γ^L we get Γ_t^{L}. Moreover, because of (8) and our observation there, $\Gamma_t^{\wedge}$ can be regarded as a sentence of $\Gamma^{\wedge}$, hence $L\Gamma_t^{\wedge}$ can be regarded as in Γ. Then since Γ_t is consistent, $\Gamma_t^{\mathrm{L}} \vdash L\Gamma_t^{\wedge}$. Thus we have $\Gamma_t^{\mathrm{L}} \wedge L\Gamma_t^{\wedge}$. By (9) and (7), we get $L\Gamma_t$.

(In fact, $\Gamma_s \wedge L\Gamma_t$ is consistent iff s and t agree on Γ^L iff $\vdash \Gamma_s \to L\Gamma_t$.) ⊣

As a consequence of this result, we call $\Gamma(\phi)$ the *strong closure of ϕ* under L.

Let ϕ be consistent. Our strategy to get a finite model for ϕ will be to carry out a filtration on $\mathcal{C}(ca)$, using $\Gamma(\phi)$. We will use the (standard) minimal filtration for both modalities.

More precisely, let $S \in \mathcal{C}(ca)$. Abusing the notation, we will write Γ_S for Γ_{1_S}, where 1_S is the characteristic function of S on Γ. Define an equivalence relation $\sim_\Gamma$ on $\mathcal{C}(ca)$ by

$$S \sim_\Gamma T \quad \text{iff} \quad \Gamma_S \ = \ \Gamma_T.$$

These are equivalent to the condition that $S \cap \Gamma = T \cap \Gamma$. Let $[S]$ denote the $\sim_\Gamma$-equivalence class of S, and let $[\mathcal{C}(ca)]$ be the set of all equivalence classes of m-theories. Note that $[\mathcal{C}(ca)]$ is finite; indeed its size is at most 2^N, where N is the size of Γ.

We define relations $\xrightarrow{L}$ and $\xrightarrow{\diamond}$ on $[\mathcal{C}(ca)]$ as follows:

$$[S] \xrightarrow{L} [T] \quad \text{iff} \quad \text{there exist } S' \in [S] \text{ and } T' \in [T] \text{ such that } S' \xrightarrow{L} T'.$$

We define $\xrightarrow{\diamond}$ on $[\mathcal{C}(ca)]$ similarly. Note that $S \xrightarrow{L} T$ implies $[S] \xrightarrow{L} [T]$, and similarly for $\xrightarrow{\diamond}$. We call the tuple

$$\langle \mathcal{C}(ca)], \xrightarrow{L}, \xrightarrow{\diamond} \rangle$$

the *quotient of* $\mathcal{C}(ca)$ *by* $\sim_\Gamma$. We shall show that this quotient is a cross axiom frame. The next two results are used in the verifications.

Proposition 2.8 *The following are equivalent:*

1. $[S] \xrightarrow{L} [T]$;
2. $\Gamma_S \wedge L\Gamma_T$ *is consistent;*
3. $\vdash \Gamma_S \to L\Gamma_T$.

Proof

(1)→(2): Choose S' and T' as above; then $\Gamma_S \wedge L\Gamma_T$ is in S'.

(2)→(3): This is from Proposition 2.7.

(3)→(1): Suppose that $\vdash \Gamma_S \to L\Gamma_T$; then in particular $L\Gamma_T \in S$, so there is a complete theory T' such that $S \xrightarrow{L} T'$, and so that T' contains $L\Gamma_S$. Of course $T' \in [T]$.

$\dashv$

Proposition 2.9 *The following are equivalent:*

1. $[S] \xrightarrow{\diamond} [T]$;
2. $\Gamma_S \wedge \diamond\Gamma_T$ *is consistent.*

Proof (1) → (2) is again by the definition of $\xrightarrow{\diamond}$ on $[\mathcal{C}(ca)]$. Going the other way, let S' be a maximal consistent set containing $\Gamma_S \wedge \diamond\Gamma_T$. Then $S' \in [S]$, and also there is some T' containing Γ_T so that $S' \xrightarrow{\diamond} T'$. It follows that $[S] \xrightarrow{\diamond} [T]$ in $[\mathcal{C}(ca)]$. $\dashv$

Lemma 2.10 *Let ϕ be any sentence, and let $\Gamma = \Gamma(\phi)$ the strong closure of ϕ under L. Let $[\mathcal{C}(ca)]$ be the quotient of $\mathcal{C}(ca)$ by $\sim_\Gamma$. Then $[\mathcal{C}(ca)]$ is a cross axiom frame.*

Proof We check that $\xrightarrow{L}$ is an equivalence relation on $[\mathcal{C}(ca)]$. Reflexivity is trivial. For symmetry, suppose that $[S]\xrightarrow{L}[T]$. Then $\Gamma_S \wedge L\Gamma_T$ is consistent. Hence also $L\Gamma_S \wedge \Gamma_T$ is consistent (since it is satisfied in any model of $\Gamma_S \wedge L\Gamma_T$). So $[T]\xrightarrow{L}[S]$. Finally, for transitivity, suppose that $[S]\xrightarrow{L}[T]\xrightarrow{L}[U]$. Then using Proposition 2.8, we get that $\Gamma_S \vdash L\Gamma_T$ and $\Gamma_T \vdash L\Gamma_U$. Hence $\Gamma_S \vdash L\Gamma_U$ and $[S]\xrightarrow{L}[U]$.

We turn to $\xrightarrow{\diamond}$. This relation is trivially reflexive on $[\mathcal{C}(ca)]$, and the only difficult step of the entire construction is to verify that $\xrightarrow{\diamond}$ is transitive. Before we do this, let us check the cross property. Suppose that

$$[S]\xrightarrow{\diamond}[T]\xrightarrow{L}[U].$$

Choose $S' \in [S]$ and $T' \in [T]$ such that $S'\xrightarrow{\diamond}T'$. By Proposition 2.8, $\Gamma_T \vdash L\Gamma_U$. So $L\Gamma_U \in T'$. This means that we can extend Γ_T to a complete theory $U' \in [U]$ such that $T'\xrightarrow{L}U'$. Since $\mathcal{C}(ca)$ satisfies the cross property, there is a complete theory v such that $s\xrightarrow{L}v\xrightarrow{\diamond}u$. Thus $[S]\xrightarrow{L}[V]\xrightarrow{\diamond}[U]$.

For transitivity of $\xrightarrow{\diamond}$, we use the characterization of Proposition 2.9. We must show that if $\Gamma_S \wedge \diamond\Gamma_T$ and $\Gamma_T \wedge \diamond\Gamma_U$ are both consistent, then so is $\Gamma_S \wedge \diamond\Gamma_U$. For this we will use a semantic argument based on the completeness of the logic of subset spaces.

Let J be a cross axiom model containing some j^* such that $j^* \models \Gamma_S \wedge \diamond\Gamma_T$, and let K be a cross axiom model containing some k^* such that $k^* \models \Gamma_T \wedge \diamond\Gamma_U$.

Consider a new model $J\oplus_\Gamma K$ defined as follows: The set of nodes of $J\oplus_\Gamma K$ is the disjoint union of the nodes of J and K. For $p, q \in J\oplus_\Gamma K$, we define $p\xrightarrow{L}q$ if this relation holds in either J or K. Further, $p\xrightarrow{\diamond}q$ holds in $J\oplus_\Gamma K$ if it holds in either J or K *or* if there are $j \in J$ and $k \in K$ such that $p\xrightarrow{\diamond}j$ in J, $k\xrightarrow{\diamond}q$ in K, and

$$\Gamma \cap th_J(j) \quad = \quad \Gamma \cap th_K(k). \tag{10}$$

An equivalent way to write (10) is $\Gamma_{th(j)} = \Gamma_{th(k)}$. (These are equivalent since sets of the form $th(j)$ are maximal.)

Note that $J\oplus_\Gamma K$ is not the same as $K\oplus_\Gamma J$. The reason is that the condition on the $\xrightarrow{\diamond}$ relation on $J\oplus_\Gamma K$ is not symmetric in J and K.

We check that $J\oplus_\Gamma K$ is a cross axiom frame. It is obvious that $\xrightarrow{L}$ is an equivalence relation on $J\oplus_\Gamma K$, and that $\xrightarrow{\diamond}$ is reflexive. For transitivity of $\xrightarrow{\diamond}$, there are only two nontrivial cases: when $j\xrightarrow{\diamond}p\xrightarrow{\diamond}k$ where $j \in J$, $k \in K$, and p is in either J or K. Suppose that $p \in K$. Then there are $c \in J$ and $d \in K$ such that $j\xrightarrow{\diamond}c$, $d\xrightarrow{\diamond}p$, and $\Gamma \cap th(c) = \Gamma \cap th(d)$. Then by transitivity within K, we have $d\xrightarrow{\diamond}k$, so the same c and d witness that $j\xrightarrow{\diamond}k$ in $J\oplus_\Gamma K$. The argument when $p \in J$ is similar.

To check that $J\oplus_\Gamma K$ satisfies the cross property, the only nontrivial case is when

$$j \quad \xrightarrow{\diamond} \quad k \quad \xrightarrow{L} \quad k',$$

where $j \in J$ and $k, k' \in K$. Let $c \in J$ and $e \in K$ be such that $j\xrightarrow{\diamond}c$, $e\xrightarrow{\diamond}k$, and $\Gamma \cap th(c) = \Gamma \cap th(e)$. Applying the cross property in K we can find some e' such that

$e \xrightarrow{L} e' \xrightarrow{\Diamond} b'$. By Proposition 2.8, $\Gamma_e \vdash L\Gamma_{e'}$. Since $e \models \Gamma_e$, there must be some $c' \in J$ such that $c \xrightarrow{L} c'$ and $\Gamma_{th(c')} = \Gamma_{th(e')}$. By the cross property of J, we can find some j' such that $j \xrightarrow{L} j' \xrightarrow{\Diamond} c'$. And c' and e' witness that $j \xrightarrow{L} j' \xrightarrow{\Diamond} k'$.

We have completed the verification that $J \oplus_\Gamma K$ is a cross axiom frame. We get a model in the obvious way, by taking the interpretation of an atomic sentence to be the union of its interpretations in J and K.

Claim For all $j \in J$ and $\psi \in \Gamma$, $j \models_K \psi$ iff $j \models_{J \oplus_\Gamma K} \psi$.

Proof By induction on $\phi \in \Gamma$. The atomic step is immediate, and the induction steps for the boolean connectives are trivial. For L, note that if $L\psi \in \Gamma$, then also $\psi \in \Gamma$. If $j \models_{J \oplus_\Gamma K} L\psi$, then there is some $p \in J \oplus_\Gamma K$ so that $p \models_J \psi$. This p must belong to J by the definition of $\xrightarrow{L}$ in $J \oplus_\Gamma K$. So the induction hypothesis applies, and $j \models_J L\psi$. Conversely, if $j \models_J L\psi$, the induction hypothesis implies that $j \models_{J \oplus_\Gamma K} L\psi$.

For $\Diamond$, assume that $\Diamond\phi \in \Gamma$. Then also $\phi \in \Gamma$. Suppose that $j \models_{J \oplus_\Gamma K} \Diamond\phi \in \Gamma$; we prove that $j \models_J \Diamond\phi$. There is some $p \in J \oplus_\Gamma K$ such that $j \xrightarrow{\Diamond} p$ and $p \models_{J \oplus_\Gamma K} \phi$. Suppose that $p \in K$ (otherwise the argument is trivial). Then there are $c \in J$ and $k \in K$ such that $j \xrightarrow{\Diamond} c$, $b \xrightarrow{\Diamond} p$, and $\Gamma \cap th(c) = \Gamma \cap th(k)$. Since $k \models_K \Diamond\phi$ and $\Diamond\phi \in \Gamma$, $c \models_J \Diamond\phi$. Then there is some $p' \in J$ such that $c \xrightarrow{\Diamond} p'$ and $p' \models_J \phi$. By induction hypothesis, $p' \models_{J \oplus_\Gamma K} \phi$. By the transitivity of $\xrightarrow{\Diamond}$ in J, $j \models_J \Diamond\phi$.

The converse assertion for $\Diamond$ is immediate from the induction hypothesis. ⊣

We now complete the proof that $\Gamma_S \wedge \Diamond\Gamma_U$ is consistent. Recall that we have $j^* \in J$ and $k^* \in K$ so that $j^* \models_J \Gamma_S$ and $k^* \models_K \Diamond\Gamma_U$. By the claim above, $j^* \models \Gamma_S$ in $J \oplus_\Gamma K$. Furthermore, a similar claim holds for K, except the proof is easier. (All arrows in $J \oplus_\Gamma K$ which begin at a node of K also end at node of K. So the inclusion of K into $J \oplus_\Gamma K$ is a bisimulation.) So $k^* \models_{J \oplus_\Gamma K} \Diamond\Gamma_U$. Moreover, let $c \in J$ be such that $j^* \xrightarrow{\Diamond} c$ and $c \models \Gamma_T$, and let $d \in K$ be such that $k^* \xrightarrow{\Diamond} d$ and $d \models_{J \oplus_\Gamma K} \Gamma_U$. Then c and k^* witness that $j^* \xrightarrow{\Diamond} d$ in $J \oplus_\Gamma K$, so that $j^* \models_{J \oplus_\Gamma K} \Diamond\Gamma_U$. Thus $\Gamma_S \wedge \Diamond\Gamma_U$ is consistent.

This completes the proof of Lemma 2.10. ⊣

At this point, we have shown that $[\mathcal{C}(ca)]$ is a cross axiom frame. We get a cross axiom model by stipulating that for atomic A,

$$[S] \models A \quad \text{iff} \quad A \in \Gamma \text{ and there is some } S' \in [S] \text{ such that } A \in S'.$$

This is well-defined, since Γ is closed under subformulas.

It is easy to check that our definition of L on $[\mathcal{C}(ca)]$ satisfies the filtration conditions

$$\frac{S \xrightarrow{L} T}{[S] \xrightarrow{L} [T]} \quad \text{and} \quad \frac{K\phi \in \Gamma,\ \Gamma_S \vdash K\phi,\ [S] \xrightarrow{L} [T]}{\Gamma_T \vdash \phi},$$

and that $\Diamond$ satisfies the analogous properties. So for all S and all $\psi \in \Gamma$,

$$S \models_{\mathcal{C}(ca)} \psi \quad \text{iff} \quad [S] \models_{[\mathcal{C}(ca)]} \psi .$$

All of these facts are standard consequences (see, e.g., [Go]) of the fact that we are dealing with the minimal filtration.

From these facts, we can complete the proof of the finite model property. Let ϕ be consistent, and let S^* be a complete theory containing ϕ. From the filtration conditions above, it follows that for all m-theories$S$, $[S] \models S \cap \Gamma$ in the finite cross axiom model $[\mathcal{C}(ca)]$. In particular, $[S^*] \models \phi$. Furthermore, we could get a bound on the size of $[S^*]$ as a computable function of ϕ. So we have proved the following:

Theorem 2.11 *Any consistent sentence ϕ has a finite cross axiom model. Since the subset space logic is complete for interpretations in cross axiom models, it is therefore decidable.*

2.4 On the Logic of Intersection Spaces

In this section, we present a few results about intersection spaces. At the present time, these results are fragmentary since we do not have completeness.

Weak Directedness Axiom $\Diamond\Box\phi \rightarrow \Box\Diamond\phi$.

Proposition 2.1 *The Weak Directedness Axiom is equivalent to the following scheme:*

$$\Diamond\Box\phi \wedge \Diamond\Box\psi \;\rightarrow\; \Diamond\Box(\phi \wedge \psi) .$$

Concerning the complete theories assuming this axiom, we also have:

(1) For all T, there is some V so that $T \xrightarrow{\Box\Diamond} V$.

(2) If $T \xrightarrow{\Box\Diamond} V$ and $T \xrightarrow{\Diamond} U$, then $U \xrightarrow{\Box\Diamond} V$.

Proof All of these assertions are standard results for a weakly-directed $S4$ modality. ⊣

Before presenting the remaining axioms for intersection spaces, we present a few consequences of our main examples.

Proposition 2.2 *Concerning spaces closed under intersection:*

(1) The subset space axioms and the Weak Directedness Axiom are not complete for intersection spaces.

(2) *The axiom* $\Box\Diamond\phi \to \Diamond\Box\phi$ *(the converse of the Weak Directedness Axiom) is not sound for intersection spaces. (Later, we show that this axiom holds in all lattices.)*

(3) *There are m-theories*$T \neq U$ *realizable in intersection spaces so that* $T \xrightarrow{\Box\Diamond} U \xrightarrow{\Box\Diamond} T$.

Proof We use the notation of Examples B and C, together with the facts about them from Section 1.3.

A look at Example C will show why the axioms so far are not complete for intersection spaces. Note first that all the m-theories realized in this model do satisfy the Weak Directedness Axiom. The reason is that up to L-equivalence, T_0 is the *unique* $\subseteq$-minimal theory of $\mathcal{X}$. From this fact it is easy to check that the Weak Directedness Axiom is satisfied in the m-theory T.

We showed already that T has no models closed under intersection.

To get an example of (2), we take Example B and the sentence $\phi = L(\textsf{first-and-}B)$. Then $c, X \models \Box\Diamond\phi$ since for all i, $(c, v_i) \models L(\textsf{first-and-}B)$. But since no c, u_i satisfies $L(\textsf{first-and-}B)$, $c, X \models \neg\Diamond\Box\phi$.

For (3), consider Example B and $T = th(c, u_0)$ and $U = th(c, u_1)$. ⊣

To summarize, the axioms of subset spaces together with Weak Directedness Axiom are not complete.

As we noted above, the Weak Directedness Axiom is unfortunately not strong enough to yield a complete axiomatization of the validities of intersection spaces. We present here an infinite family of new axioms, which we call the Sliding Axioms. Since they are a rather complicated scheme, we present a simple case first. We have not been able to prove completeness for these axioms, only soundness together with the fact that no Sliding Axiom follows from the previous axioms.

Sliding Axiom, Preliminary Form

$$\Big(\bigwedge_{i\leq n} \Diamond\phi_i \;\wedge\; \Box\Diamond L\chi\Big) \quad\to\quad L\Big(\bigwedge_{i\leq n} \Diamond(L\phi_i \wedge \Diamond\chi)\Big)\ .$$

We check that this axiom is sound for spaces closed under intersections. For suppose x, w satisfies the antecedent. Then for $i \leq n$, there are $u_i \subseteq w$ so that $x, u_i \models \phi_i$. Let $v = \bigcap u_i$; $v \in \mathcal{O}$ by hypothesis. Then there is a subset $v' \subseteq v$ and some $y \in v'$ so that $y, v' \models \chi$. But

$$y, w \models \bigwedge_{i\leq n} \Diamond(L\phi_i \wedge \Diamond\chi)\ .$$

Thus x, w satisfies the conclusion of the axiom.

Let $\mathcal{P} = \langle P, \leq, \bot, \top \rangle$ be a finite, bounded, acyclic poset, and let $\phi : \mathcal{P} \to \mathcal{L}$. Define by downward recursion on $p \in \mathcal{P}$ sentences $[\mathcal{P}, \phi]_p$ by

$$[\mathcal{P}, \phi]_p \quad = \quad \phi_p \ \wedge \ \Diamond \bigwedge_{q > p} [\mathcal{P}, \phi]_q \ .$$

For example, suppose that $\mathcal{X} = \langle X, \mathcal{O} \rangle$ is a subset space, $x \in X$, and $u : \mathcal{P} \to \mathcal{O}$ is an antitone map (so that $x \in u_p$ for all p). Let ϕ_p be any sentence so that $x, u_p \models \phi_p$. Then for all p, $x, u_p \models [\mathcal{P}, \phi]_p$. It will be very instructive for the reader to investigate the converse of this observation. Suppose $\mathcal{P}$ is a poset, and ϕ is such that $x, X \models [\mathcal{P}, \phi]_\bot$. Is it possible to find an antitone map $p \mapsto u_p$ so that for all p, $x, u_p \models \phi_p$? The answer is that it is not generally possible. The problem is that we might have q above both r and s, and also $u_r \models \phi_r$ and $u_s \models \phi_s$. We know that there is some $u_{q,r} \subseteq u_r$ so that $u_{q,r} \models \phi_q$. Also, there is some $u_{q,s} \subseteq u_s$ so that $u_{q,s} \models \phi_q$. But what we need is some $u_q \subseteq u_r \cap u_s$ with this property. To be sure u_q exists, we need the original space to satisfy something stronger than $[\mathcal{P}, \phi]_\bot$. And for this, we need a bit of notation.

Let $lin(P) \subseteq P$ be the set of elements of P whose lower sets are linearly ordered. Each $p \in lin(P)$ has a unique *predecessor* q; that is $q < p$ is unique so that there is no $q < r < p$. Similarly, let $nonlin(P) = P \setminus lin(P)$. Note that $lin(P)$ is closed downward, and $nonlin(P)$ is closed upward in $\mathcal{P}$.

A poset is *non-linear* if $\top \in nonlin(P)$. The non-linear case is the interesting one in the results below because when the order is linear, the results below have easier proofs.

Let S be a finite set of sentences, and then let $\mathcal{B}(S)$ be the set of complete boolean combinations of sentences from S. If $\phi : \mathcal{P} \to \mathcal{L}$, $\beta : \mathcal{P} \to \mathcal{B}(\mathcal{S})$ and $\chi \in \mathcal{L}$, then we define $\phi^\beta : P \to \mathcal{L}$ by

$$(\phi^\beta)_p \quad = \quad \begin{cases} \chi \ \wedge \ \beta_\top & \text{if } p = \top \text{ and } \phi_\top = L\chi \\ L\phi_p \ \wedge \ \beta_p & \text{otherwise} \end{cases} \quad .$$

Intersection Axiom II (Sliding Axiom) For each finite, bounded, acyclic, non-linear poset $\mathcal{P}$, each finite set $S \subseteq \mathcal{L}$, and each $\phi : \mathcal{P} \to \mathcal{L}$ so that $\phi_\top$ is an L-sentence $L\chi$,

$$[\mathcal{P}, \phi]_\bot \ \wedge \bigwedge_{p \in nonlin(P)} \Box\Diamond\phi_p \quad \to \quad L \bigvee_{\beta : \mathcal{P} \to \mathcal{B}(S)} [\mathcal{P}, \phi^\beta]_\bot \ .$$

Proposition 2.3 *The Sliding Axiom is sound for intersection spaces.*

Proof Fix an intersection space $\mathcal{X} = \langle X, \mathcal{O} \rangle$, and consider an instance of the Sliding Axiom. Assume x, X satisfies the antecedent. Then we define an antitone map $u : \mathcal{P} \to \mathcal{O}$ so that $x \in u_p$ for all p, and so that $x, u_p \models [\mathcal{P}, \phi]_p$. The definition is by recursion on $\mathcal{P}$. Let $u_\bot = X$. Suppose we are given u_q for $q < p$. There are two cases: either $p \in lin(P)$ or $p \in nonlin(P)$. First we define u_p for all $p \in lin(P)$. Then the elements below p are a linear order, and in particular, there is a unique predecessor q of p. Then since $x, u_q \models [\mathcal{P}, \phi]_q$, there is some $u_p \subseteq u_q$ so that $x, u_p \models [\mathcal{P}, \phi]_p$. The point is that since q is the predecessor of q, we have that $u_r \supseteq u_p$ whenever $r \leq p$.

At this point we have u_p for all $p \in lin(P)$. We then turn to those $p \in nonlin(P)$. We define u_p so that $x, u_p \models \phi_p$. (Once the definition is complete, it then follows that $x, u_p \models [\mathcal{P}, \phi]_p$.) We first consider $v_p = \bigcap\{u_q : q > p\}$. Since $x, X \models \Box\Diamond\phi_p$, let $u_p \subseteq v_p$ be such that $x, u_p \models \phi_p$.

Since $x, u_\top \models \phi_\top = L\chi$, let $y \in u_\top$ be such that $y, u_\top \models \chi$. Note that $y \in u_p$ for all p, so we may let let $\beta(p)$ be the unique boolean combination from the given finite set S so that $y, u_p \models \beta(p)$. Then $y, X \models [\mathcal{P}, \phi^{\beta,\chi}]_\bot$. In this way, x, X satisfies the conclusion of this instance of the Sliding Axiom. $\dashv$

3 The Logic of Lattice Spaces, Complete Lattice Spaces, and Topological Spaces

We turn to lattice spaces, those which are closed under finite unions and intersections. We are even interested in the smaller classes of topological spaces and complete lattice spaces. As it happens, though, the axioms for finite lattices turn out to be complete for the smaller classes. This is in contrast to the case for intersection spaces, where the complete intersection spaces satisfy the law $\Box\Diamond\phi \to \Diamond\Box\phi$ (see the discussion just before Corollary 3.11). The spaces closed under finite intersections do not necessarily satisfy this law, as shown in Proposition 2.2. It is not at all obvious that lattice spaces satisfy this axiom, but it will follow from our results below.

Our main results in this section are the completeness and decidability of topologic (Theorems 3.10 and 3.19). The latter result was first proved by Georgatos [G2] by a different argument. Our method gives a connection of topologic to much earlier work on topology and modal logic due to McKinsey. Indeed, we believe that our proof of McKinsey's Theorem 3.6 is new, and that result is the basis of our completeness result for topologic.

3.1 Bi-persistent Sentences and the Prime Normal Form Lemma

The work in this section is essentially a simplification of the construction from Konstantinos Georgatos' Ph.D. thesis [G1] The overall goal is a normal form theorem which plays a key role in the completeness theorem for topologic.

Definition A sentence ϕ is *bi-persistent* if $\vdash \Diamond\phi \to \Box\phi$. Semantically, this means that satisfaction of ϕ in a world x, u depends only on the point x. So we say ϕ is *bi-persistent for intersection spaces* if for all intersection spaces $\mathcal{X} = \langle X, \mathcal{O}\rangle$ and all $p \in u \in \mathcal{O}$, if $p, u \models \phi$, then $p, X \models \Box\phi$. We define *bi-persistence for lattice spaces* and *bi-persistence for topological spaces* similarly.

Definition Let Π be the smallest set of sentences containing the atomic sentences, and closed under boolean operations and the operators $\Diamond K$ and $\Box L$.

Proposition 3.1 (Georgatos [G2]) *All sentences in Π are bi-persistent on all subset spaces satisfying the intersection axiom. In particular, all sentences in Π are bi-persistent for intersection spaces.*

Proof By induction on $\phi \in \Pi$. The atomic sentences are bi-persistent by the axiom $(A \to \Box A) \wedge (\neg A \to \Box\neg A)$. The negation of a bi-persistent sentence is easily seen to be bi-persistent. Suppose ϕ and ψ are both bi-persistent. Assuming $\Diamond(\phi \wedge \psi)$ gives $\Diamond\phi \wedge \Diamond\psi$; whence $\Box\phi \wedge \Box\psi$. So $\Box(\phi \wedge \psi)$.

Finally, assume that ϕ is bi-persistent. To show that $\Diamond K\phi$ is bi-persistent, assume $\Diamond\Diamond K\phi$. Then $\Diamond K\phi$, so $\Diamond K\Diamond\phi$. Using bi-persistence, we have $\Diamond K\Box\phi$. By the cross axiom, $\Diamond\Box K\phi$. Now by the Weak Directedness Axiom, $\Box\Diamond K\phi$. $\dashv$

Remarks It is easy to check that the Weak Directedness Axiom is needed in order to prove Proposition 3.1. For example, consider a subset space with universe $X = \{a, b, c\}$, opens $\{a, b\}$ and $\{a, c\}$, and such that the interpretation of the atomic A is $\{a, b\}$. Then

$$a, X \models (\Diamond\Diamond KA) \wedge (\neg\Box\Diamond KA) .$$

So the sentence $\Diamond KA$ need not be bi-persistent in spaces not closed under intersections.

Second, the converse of Proposition 3.1 is Theorem 3.15 below. That result is a consequence of the completeness theorem for lattice spaces.

Definition Let Σ be the closure under $\wedge$ of the set of sentences of the form $L\pi$ or $K\pi$, for $\pi \in \Pi$. Since K distributes over $\wedge$, any sentence σ in Σ is of the form $K\pi \wedge \bigwedge_i L\pi'_i$. A sentence ϕ is in *prime normal form (pnf)*, if ϕ is of the form $\pi \wedge \sigma$ where $\pi \in \Pi$ and $\sigma \in \Sigma$. Finally, ϕ is in *normal form (nf)*, if ϕ is a disjunction of sentences in pnf.

Semantically, satisfaction of a Σ sentence depends only on the neighborhood, not on the point. Thus, Σ is dual to Π. Our immediate goal is a canonical form for sentences (Lemma 3.3), which asserts that on set lattices, the language $\mathcal{L}$ is essentially just the boolean closure of $\Sigma \cup \Pi$. We first need the following lemma.

Lemma 3.2 *If* $\pi, \pi', \pi_1, \ldots, \pi_n \in \Pi$ *then*

1. $\vdash L\Diamond(K\pi' \wedge \bigwedge_i L\pi_i) \leftrightarrow \bigwedge_i L\Diamond(K\pi' \wedge \pi_i)$;
2. $\vdash \Diamond[\pi \wedge K\pi' \wedge \bigwedge_i L\pi''_i] \leftrightarrow [\Diamond(\pi \wedge K\pi') \wedge \bigwedge_i L\Diamond(K\pi' \wedge \pi''_i)]$.

Proof The implications $\rightarrow$ are trivial; the converse directions can be argued semantically or by appealing to the Union Axiom and bi-persistence. For (1), suppose $x, U \models \bigwedge_i L\Diamond(K\pi' \wedge \pi_i)$; then there are worlds $x_i \in U_i \subseteq U$ such that $x_i, U_i \models K\pi' \wedge \pi_i$. Taking the union we get a neighborhood $U' = \bigcup_i U_i \subseteq U$ and since the πs are bi-persistent, satisfaction isn't affected by the ambient neighborhood: $x_i, U' \models K\pi' \wedge \pi_i$.

We just sketched a semantic proof of (1) so we will give a formal proof only of the right-to-left direction of (2). As a preliminary, note that if ρ is bi-persistent, then $\vdash K\Diamond\rho \rightarrow K\rho$. Also, $\vdash L\Diamond\rho \rightarrow L\rho$, and hence $\vdash L\Diamond L\rho \rightarrow L\rho$.

Assume $\Diamond(\pi \wedge K\pi') \wedge \bigwedge_i L\Diamond(K\pi' \wedge \pi''_i)$. By iterated use of the Union Axiom, we have

$$\Diamond\Big[\Diamond(\pi \wedge K\pi') \quad \wedge \quad \bigwedge_i L\Diamond(K\pi' \wedge \pi''_i) \quad \wedge \quad K\Diamond L(\pi \wedge K\pi' \vee \bigvee_i (K\pi' \wedge \pi''_i))\Big] .$$

Note that

$$\vdash K\Diamond L(\pi \wedge K\pi' \vee \bigvee_i (K\pi' \wedge \pi_i'')) \rightarrow K\Diamond\pi' \ ,$$

and $\vdash K\Diamond\pi' \rightarrow K\pi'$. Also, $\vdash L\Diamond\pi_i'' \rightarrow L\pi_i''$ for all i. Putting these facts together and using propositional reasoning, we get

$$\Diamond[\pi \quad \wedge \quad K\pi' \quad \wedge \quad \bigwedge_i L\pi_i''] \ .$$

This is the left side of (2). ⊣

Lemma 3.3 (Normal Form Lemma) *(Georgatos [2]) Any sentence is equivalent to a sentence in normal form.*

Proof By induction on sentences built from atomic sentences using $\vee$, $\neg$, L, and $\Diamond$. The cases of atomic sentences and $\phi \vee \psi$ are trivial. For negation, we use De Morgan's Laws and the lattice laws of conjunction and disjunction, together with the fact that Π is closed under negation.

Assume that ϕ is equivalent to a disjunction of pnf's. We show that $L\phi$ has the same property. Since $\vdash L(\phi_1 \vee \phi_2) \leftrightarrow (L\phi_1 \vee L\phi_2)$, we may assume without loss of generality that ϕ itself is in pnf. By induction hypothesis, we have π, π', and π_i'' from Π so that $\vdash \phi \leftrightarrow (\pi \wedge K\pi' \wedge \bigwedge_i L\pi_i'')$. Then

$$\vdash L\phi \leftrightarrow [K\pi' \wedge (L\pi \wedge \bigwedge_i L\pi_i'')].$$

Finally, assume that ϕ is equivalent to a disjunction of pnf's; we prove this for $\Diamond\phi$. As in the case of L, we may assume that ϕ itself is in pnf. By induction hypothesis, we have π, π', and π_i'' from Π so that $\vdash \phi \leftrightarrow (\pi \wedge K\pi' \wedge \bigwedge_i L\pi_i'')$. By Lemma 3.2, part 2, $\Diamond\phi$ is equivalent to $\Diamond(\pi \wedge K\pi') \wedge \bigwedge_i L\Diamond(K\pi' \wedge \pi_i'')$. Now $\Diamond(\pi \wedge K\pi')$ is equivalent to $\pi \wedge \Diamond K\pi'$; this belongs to Π. Bi-persistence also implies that each $L\Diamond(K\pi' \wedge \pi_i'')$ is equivalent to $L(\Diamond K\pi' \wedge \pi_i'')$. In this way, $\Diamond\phi$ is equivalent to a conjunction of a sentence in Π with a sentence in Σ. ⊣

Lemma 3.3 provides a sort of "orthogonal decomposition" of the language into Π and Σ components. We discuss this in the next section.

3.2 McKinsey's Theorem

J. C. C. McKinsey [McK] initiated a study of the relation of $S4$ to topology[1] He used the symbols C and I (for *closure* and *interior*) instead of $\Diamond$ and $\Box$. To avoid confusion we will

[1]We should mention that our discussion of McKinsey's work is necessarily anachronistic and revisionist, since he worked directly on the basis of Lewis' algebraic formulation of modal logic, before Kripke semantics.

follow him and use C and I for, respectively, the existential and universal modalities of $S4$. He noted that the interior operator on a topological space has $S4$-like properties. He also proved a completeness result and a finite model property for his topological interpretation of $S4$. We review this work below, and we also present proofs of McKinsey's theorems. As it happens, our completeness and finite model results for topologic can be proved from McKinsey's theorems.

McKinsey notes that the logic of $S4$ corresponds to laws concerning the boolean operations and the operation of interior on a topological space. That is, suppose that $\langle X, \mathcal{O}\rangle$ is a topological space, and consider a map i of atomic sentences A_i of $\mathcal{L}(S4)$ into the subsets of X. Then i extends to all of $\mathcal{L}(S4)$ by interpreting negation as complement relative to X, conjunction as intersection, and C and the closure operator. With the space X fixed, we write $x \models_{McK} \phi$ iff $x \in i(\phi)$. If $S \cup \{\phi\} \subseteq \mathcal{L}(S4)$, we write $S \models_{McK} \phi$ iff for all $\langle X, \mathcal{O}\rangle$ and all i, $\bigcap\{i(\psi) : \psi \in S\} \subseteq i(\phi)$.

We want to relate this semantics of $S4$ to the standard Kripke semantics on the one hand, and to the semantics of topologic on the other.

Let $(X, \leq)$ be a Kripke frame satisfying $S4$. (We have written the accessibility relation with the symbol for a preorder, since it necessarily is reflexive and transitive.) Consider the Alexandrov topology on this frame, where the opens are the sets closed upwards in the order. In this way, we associate a topological interpretation to each Kripke frame interpretation of $S4$. (Actually, this gives a special kind of space: the opens are closed under arbitrary intersections. All of our results concerning topological spaces hold for the smaller class of subset spaces closed under all unions and intersections.)

Proposition 3.4 *For all $S4$ frames $(X, \leq)$, all $x \in X$, and all $\phi \in \mathcal{L}(S4)$,*

$$x \models_{S4} \phi \quad \textit{iff} \quad x \models_{McK} \phi \ .$$

In other words, the two semantics for $S4$ agree.

Proof By induction on $\phi \in \mathcal{L}(S4)$. The case of the atomic sentences is trivial, as are the induction steps for the boolean connectives.

Assume the lemma for ϕ, and consider $C\phi$. If $x \models_{S4} C\phi$, then there is some $y \geq x$ so that $y \models_{S4} \phi$. By induction hypothesis, $y \models_{McK} \phi$. But every neighborhood containing x also contains y, so x is in the closure of $\{z : z \models_{McK} \phi\}$. Thus $x \models_{McK} C\phi$. Conversely, if $x \models_{McK} C\phi$, then every neighborhood of x contains a point y so that $y \models_{McK} \phi$. In particular, $\uparrow x$ contains such a y. This gives some $y \geq x$ which (by the induction hypothesis) has the property that $\models_{S4} \phi$. Hence $x \models_{S4} C\phi$. $\dashv$

To relate the McKinsey semantics to our semantics of topologic, we define a map

$$^* : \mathcal{L}(S4) \to \Pi$$

from $\mathcal{L}(S4)$ onto Π by recursion:

$$\begin{array}{lcl} A^* & = & A \\ (\phi \wedge \psi)^* & = & \phi^* \wedge \psi^* \\ (\neg\phi)^* & = & \neg(\phi^*) \\ (I\phi)^* & = & \Diamond K \phi^* \end{array}$$

Note that we have formulated $\mathcal{L}(S4)$ using a modal operator I. We could have also used $\Box$; the point only is that the universal modality corresponds to $\Diamond K$ and not to $\Box L$, since for $\phi \in \Pi$, $\vdash \Diamond K\phi \to \phi$, but $\not\vdash \Box L\phi \to \phi$. (That is, the interior of a set is a subset of the set; not so the closure.) When McKinsey speaks of interpreting a sentence ϕ of $\mathcal{L}(S4)$ in a topological space, then in our terms he is using ϕ^* and the semantics of topologic.

Proposition 3.5 *For all ϕ in $\mathcal{L}(S4)$, and all topological spaces $\mathcal{X} = \langle X, \mathcal{O}\rangle$,*

$$x \models_{McK} \phi \quad \textit{iff} \quad x, X \models_{top} \phi^* \ .$$

Proof By induction on $\phi \in \mathcal{L}(S4)$. The case of the atomic sentences is trivial, as are the induction steps for the boolean connectives.

Assume the lemma for ϕ, and consider $I\phi$. If $x \models_{McK} I\phi$, then x belongs to the interior of $\{y \in X : y \models_{McK} \phi\}$. Thus there is an open u so that for all $y \in u$, $y \models_{McK} \phi$. By induction hypothesis, we see that for all $y \in u$, $(y, X) \models_{top} \phi^*$; hence for all $y \in u$, by bi-persistence $(y, u) \models_{top} \phi^*$. Therefore, $x, X \models_{top} \Diamond K\phi^*$.

The converse is proved the same way. ⊣

We might, somewhat tendentiously, say that the fact that all of the sentences ϕ^* are bi-persistent means that the McKinsey semantics is not using the topology in a very interesting way.

This map $-^*$ is a bijection of $\mathcal{L}(S4)$ onto Π, and we write its inverse as $-_* : \Pi \to \mathcal{L}(S4)$. Of course, the image of $-^*$ is only Π, not the full language $\mathcal{L}$ of topologic. We do not have a nice map of $\mathcal{L}$ into $\mathcal{L}(S4)$, but to some extent the Normal Form Lemma 3.3 will help us to get around this difficulty. We extend $-^*$ and $-_*$ to maps on sets by, e.g., $S^* = \{\phi^* : \phi \in S\}$.

Recall that the *canonical Kripke frame* for $S4$ is the set of maximal consistent theories of $S4$ with the binary relation $\xrightarrow{I}$. This relation is a preorder $\leq_I$ on $\mathcal{TH}(S4)$. We obtain a Kripke structure by associating to the atomic sentence A the set $\{x : A \in x\}$. We regard the Kripke structure for $\mathcal{L}(S4)$ as a topological space, using the Alexandrov topology of upper closed sets. Let $\mathcal{C}(S4)$ denote this structure. So $\mathcal{C}(S4)$ denotes either a Kripke frame or a topological space, and the context should clarify this. Either way, $\mathcal{C}(S4)$ is the *canonical interpretation* of $\mathcal{L}(S4)$.

Theorem 3.6 (McKinsey) *The interpretation of $S4$ in topological spaces is complete. That is, for every maximal consistent subset T of $S4$, there is a topological space $\mathcal{X} = \langle X, \mathcal{O}\rangle$*

and some $x \in X$ so that $x, X \models_{McK} T$. Moreover, if ϕ is a satisfiable sentence of $\mathcal{L}(S4)$, then ϕ is satisfiable in a finite topological space.

Proof Let $\mathcal{C}(S4)$ be the canonical Kripke structure/topological space for $S4$. By the Truth Lemma for $S4$, the theory of T in $\mathcal{C}(S4)$ is T itself. But by Proposition 3.4, this is the same as the theory of the point T, when we take the topological interpretation. This implies the completeness result.

For the finite model property, we use the finite model property for the Kripke structure interpretation of $S4$, together with Proposition 3.4 once again. $\dashv$

3.3 Completeness of Topologic

In this section, we use McKinsey's Theorem to prove completeness of topologic.

We shall work with the *canonical model* $\mathcal{C}(top)$ of topologic, defined as follows: Let $\mathcal{TH}(top)$ be the set of maximal consistent subsets of Π. That is, the subsets $x \subseteq \Pi$ which are consistent in topologic and for which there are no consistent y with $x \subset y \subseteq \Pi$. For each maximal consistent $T \subseteq \mathcal{L}$, $T \cap \Pi \in \mathcal{TH}(top)$. (The converse will follow from Lemma 3.8.) We shall use the fact that $\xrightarrow{\Box L}$ is a preorder $\leq_{\Box L}$ on $\mathcal{TH}(top)$.

Once again, we consider the Alexandrov topology on $(\mathcal{TH}(top), \leq_{\Box L})$. That is, we take $\mathcal{O}$ to be the collection of *upward-closed* subsets of $\mathcal{TH}(top)$, and then let

$$\mathcal{C}(top) \quad = \quad \langle \mathcal{TH}(top), \mathcal{O}, \alpha \rangle,$$

where $\alpha(A) = \{x \in \mathcal{TH}(top) : A \in x\}$.

Theorem 3.7 *The maps $-^*$ and $-_*$ induce inverse isomorphisms of subset spaces $-_* : \mathcal{C}(top) \to \mathcal{C}(S4)$ and $-^* : \mathcal{C}(S4) \to \mathcal{C}(top)$.*

Proof First we show that if $S \vdash_{S4} \phi$, then $S^* \vdash_{int} \phi^*$. (That is, the deduction can be carried out using the basic logic of subset spaces together with the Axiom of Intersection.) To do this, we first show by induction on proofs in $S4$ that if $\vdash_{S4} \psi$, then $\vdash_{int} \psi^*$. This is by induction on $S4$ proofs. If ϕ is an instance of a propositional tautology, then so is ϕ^*, so $\vdash_{set\ spaces} \phi^*$. The first interesting case concerns the normality condition, say in the form

$$\vdash_{S4} ((I\phi) \wedge (I\psi)) \to I(\phi \wedge \psi) .$$

In this case, note that according to the Axiom of Intersection,

$$\vdash_{int} ((\Diamond K\phi^*) \wedge (\Diamond K\psi^*)) \to \Diamond K(\phi^* \wedge \psi^*) .$$

This implication uses the fact that sentences in the image of the * map are bi-persistent, and also the Weak Directedness Axiom. Next, we consider the axioms $I\phi \to \phi$ and $I\phi \to II\phi$.

These correspond to the subset space deductions $\Diamond K\phi^* \to \phi^*$ and $\Diamond K\phi^* \to \Diamond K\Diamond K\phi^*$. (For this last fact, note that for all χ, if $K\chi$, then $\Diamond K\chi$. So $KK\chi \to K\Diamond K\chi$. Thus $\Diamond K\chi \to \Diamond K\Diamond K\chi$.)

To conclude this part of the proof, note that if $S \vdash_{S4} \phi$, then $S^* \vdash_{int} \phi^*$ as well.

It follows from this that if $x \subseteq \Pi$ is consistent in topologic, so is $x_* \subseteq \mathcal{TH}(S4)$. Then this same result holds for the property of maximal consistency.

We regard $-_*$ as a map $-_* : \mathcal{C}(top) \to \mathcal{C}(S4)$. To show that $-_*$ is surjective, let S be a maximal consistent subset of $\mathcal{L}(S4)$. S is a point in the the topological space $\mathcal{C}(S4)$, and the theory of S in the McKinsey semantics is S itself. So the theory of S in the topologic semantics is S^*. Thus S^* is maximal consistent in topologic. So $(S^*)_* = S$ is maximal consistent in $S4$.

We now know that $-^*$ and $-_*$ are bijections between the points of $\mathcal{C}(S4)$ and $\mathcal{C}(top)$. It is easy to check that $x \leq_{\Box L} y$ in $\mathcal{C}(top)$ iff $x_* \leq_I y_*$ in $\mathcal{C}(S4)$. Thus the two maps are also bijections between the collections of opens. Thus they are isomorphisms of subset spaces. ⊣

Proposition 3.8 *For all $(x,u) \in \mathcal{C}(top)$, $th_{\mathcal{C}(top)}(x,u) \supseteq x$.*

Proof Note that x_* is a point of $\mathcal{C}(S4)$, so by the Truth Lemma for that structure, the $S4$-theory of x_* in $\mathcal{C}(S4)$ is x_*. So in $\mathcal{C}(top)$, x satisfies all of the sentences in $(x_*)^* = x$. ⊣

Definition For any open set u in $\mathcal{C}(top)$, and any $x \in u$, let

$$\begin{array}{rcl} S(u) & = & \{L\pi : \pi \in \bigcup u\} \cup \{K\pi : \pi \in \bigcap u\} \\ \overline{(x,u)} & = & \{\phi : x \cup S(u) \vdash_{top} \phi\} \end{array}$$

Lemma 3.9 (The Truth Lemma for $\mathcal{C}(top)$) *For all $x \in u$, and all $\phi \in \mathcal{L}$,*

$$x, u \models_{\mathcal{C}(top)} \phi \quad \textit{iff} \quad \phi \in \overline{(x,u)} \ .$$

Proof Let $th(x,u)$ be the (topologic) theory of (x,u) in $\mathcal{C}(top)$. By Proposition 3.8, $x \subseteq th(x,u)$. We next check that that $S(u) \subseteq th(x,u)$ as well. To see this, suppose that $L\pi \in S(u)$ because some $y \in u$ contains π. Then since $y \subseteq th(y,u)$, $L\pi \in th(x,u)$. Finally, suppose that $K\pi \in S(u)$. Then again, for all $y \in u$, $\pi \in y \subseteq th(y,u)$. Thus $K\pi \in th(x,u)$.

Since $th(x,u)$ is closed under deduction, $\overline{(x,u)} \subseteq th(x,u)$. Thus $\overline{(x,u)}$ is consistent.

For all $\pi \in \Pi$, either there is some $x \in u$ so that $\pi \in x$, or else for all $x \in u$, $\neg\pi \in x$. Thus either $L\pi$ or $K\neg\pi$ belongs to $S(u) \subseteq \overline{(x,u)}$. Since $\overline{(x,u)}$ is consistent, exactly one of these holds. Similarly, for all $\pi \in \Pi$, exactly one of π or $\neg\pi$ belongs to $\overline{(x,u)}$.

It follows that if ψ is any boolean combination of sentences of the form π, $L\pi$, or $K\pi$, that exactly one of ψ or $\neg\psi$ belongs to $\overline{(x,u)}$. (The proof of this is by induction on the heights of boolean combinations, using de Morgan's laws.) By Georgatos' Prime Normal Form Lemma, every sentence is equivalent in topologic to a boolean combination of sentences of these forms. So $\overline{(x,u)}$ is maximal consistent. Therefore, $\overline{(x,u)} = th(x,u)$. $\dashv$

Theorem 3.10 *The axioms of topologic are complete for the topological interpretation. In fact, they are complete for interpretations in complete lattices.*

Proof We need to show that every maximal consistent subset T of topologic has a model. Let $x_T = T \cap \Pi$; x_T is a maximal consistent Π-theory and is thus a point in $\mathcal{C}(top)$. Let

$$u_T \quad = \quad \{y \in \mathcal{C}(top) : T \xrightarrow{L} y\} \ .$$

($T \xrightarrow{L} y$ means that for all $\pi \in y$, $L\pi \in T$.)

We claim first that $x \in u_T$; that is, that for all $\pi \in \Pi \cap T$, $L\pi \in T$. This follows from the fact that deductively closed sets are closed under L.

We next show that u_T is an open set in $\mathcal{C}(top)$. Suppose that $T \xrightarrow{L} y \xrightarrow{\Box L} z$. We show that $T \xrightarrow{L} z$. Let $\pi \in z$. Then $\Box L\pi \in y$, so $L\Box L\pi \in T$. Thus $L\pi \in T$.

By the Truth Lemma, $th(x_T, u_T) = \overline{(x_T, u_T)}$. Thus $\overline{(x_T, u_T)}$ is maximal consistent. To show that $\overline{(x_T, u_T)} = T$, it is sufficient to show that $x_T \cup S(u_T) \subseteq T$. Clearly $x_T = T \cap \Pi \subseteq T$. If $L\pi \in S(u_T)$, then by definition of u_T, $L\pi \in T$. Suppose that $K\pi \in S(u_T)$; we show that $K\pi \in T$. For if not, then $L\neg\pi \in T$. Let U be a maximal consistent set so that $T \xrightarrow{L} U$ and $\neg\pi \in U$. Let $y = U \cap \Pi$. Then y is a point of $\mathcal{C}(top)$, and $y \in u_T$ by definition of u_T. But then $K\pi \notin S(u_T)$, and this is a contradiction. This completes the verification that $x_T \cup S(u_T) \subseteq T$. $\dashv$

The fact that the canonical model for the lattice space logic is a *complete* lattice means that the logic must prove all valid sentences for the smaller class. Among these are those of the following scheme:

Complete Intersection Axiom $\Box\Diamond\phi \to \Diamond\Box\phi$. Equivalently, $\Diamond(\Box\phi \vee \Box\neg\phi)$.

The Complete Intersection Axiom is sound for complete intersection spaces. To see this, consider a point x in some space. Let u_x be the intersection of all opens containing x. Then $x, u_x \models (\Box\phi \vee \Box\neg\phi)$. On the other hand, Proposition 2.2 shows that it is not sound for intersection spaces. Intuitively, this is because closure under finite intersections is not enough to guarantee that the neighborhoods of a point eventually stabilize on ϕ or on $\neg\phi$. The interesting point is that this axiom is sound for lattice spaces.

Corollary 3.11 *For all ϕ, $\vdash \Box\Diamond\phi \to \Diamond\Box\phi$. That is, each Complete Intersection Axiom is provable from the axioms of lattice spaces.*

Proof We prove that each instance of the axiom is valid. Let $\langle X, \mathcal{O}\rangle$ be a lattice space, and let $x \in u \in \mathcal{O}$. Let $T = th_{\mathcal{X}}(x, u)$. Let $y, v \in \mathcal{C}(top)$ be such that $th_{\mathcal{C}(top)}(y, v) = T$. Then since the canonical model is closed under all intersections, each Complete Intersection Axiom belongs to $th_{\mathcal{C}(top)}(y, v) = T$. ⊣

We can also look at this result purely semantically. For each space $\langle X, \mathcal{O}\rangle$ and each x, we have $x, X \models \Diamond(\Box\phi \vee \Box\neg\phi)$. Thus there is some open u containing x so that either $x, u \models \phi$ or $x, u \models \neg\phi$. We call this fact the *minimal neighborhood property*. We do not know a completely elementary proof that it holds on lattice spaces, and we also do not know an elementary syntactic proof of Corollary 3.11.

3.4 Further Results on Bi-persistent Sentences

Theorem 3.12 shows that the embedding $-^*$ of $S4$ into Π is *faithful* in terms of proofs. That is, all theorems of topologic of the form $S \vdash \pi$, where $S \cup \{\pi\} \subseteq \Pi$, even those whose proofs use the cross axiom or the union axiom, could be obtained simply as translations of $S4$ proofs.

In a different direction, we also show that the class Π contains all sentences which are bi-persistent on all finite lattice spaces.

Theorem 3.12 *For all S and ϕ in $\mathcal{L}(S4)$,*

$$S \vdash_{S4} \phi \quad \textit{iff} \quad S^* \vdash_{top} \phi^* \quad \textit{iff} \quad S^* \vdash_{int} \phi^* .$$

Proof We showed in the proof of Theorem 3.7 that if $S \vdash_{S4} \phi$, then $S^* \vdash_{int} \phi^*$. It is trivial that if $S^* \vdash_{int} \phi^*$, then $S^* \vdash_{top} \phi^*$.

Conversely, suppose that $S \not\vdash_{S4} \phi$. Then $S \cup \{\neg\phi\}$ is consistent. The map $-^*$ preserves consistency, so $S^* \cup \{\neg\phi^*\}$ is consistent in topologic; thus $S^* \not\vdash_{top} \phi^*$. ⊣

Corollary 3.13 *If $S \subseteq \Pi$, $\pi \in \Pi$, and $S \vdash_{top} \pi$, then $S \vdash_{int} \pi$.*

Proof If $S \vdash_{top} \pi$, then by Theorem 3.12, $S_* \vdash_{S4} \pi_*$. Then $S \vdash_{int} \pi$. ⊣

This result can also be proved semantically:

Proposition 3.14 *Let $T \subseteq \Pi$. If T is satisfiable in an intersection space, then T is satisfiable in a complete lattice space.*

Proof Suppose $\mathcal{X} = \langle X, \mathcal{O}_1 \rangle$ is an intersection space such that $x, X \models T$. Then let $\mathcal{O}_2$ be the set of unions of members of $\mathcal{O}_1$. Then $\langle X, \mathcal{O}_2 \rangle$ is a topological space. Note that $\mathcal{O}_1 \subseteq \mathcal{O}_2$, but that for all $u \in \mathcal{O}_2$ there is some $v \in \mathcal{O}_1$ such that $v \subseteq u$. It is easy to check by induction on $\pi \in \Pi$ that $x, X \models_1 \pi$ iff $x, X \models_2 \pi$. So T is satisfiable in $\langle X, \mathcal{O}_2 \rangle$. $\dashv$

To conclude matters, we prove a kind converse of Proposition 3.1; all sentences which are bi-persistent for finite lattices belong to Π.

Theorem 3.15 *If ϕ is bi-persistent for finite lattice spaces, then there is some $\pi \in \Pi$ so that $\vdash_{top} \psi \leftrightarrow \pi$.*

Proof Suppose that ϕ is bi-persistent. Then by the Normal Form Lemma, $\vdash_{top} \phi \leftrightarrow \bigvee_i \phi_i$, where each ϕ_i is in pnf; and we may assume without loss of generality that each ϕ_i is consistent. Furthermore for each i write $\phi_i = \pi_i \wedge \sigma_i$, where $\pi_i \in \Pi$ and $\sigma_i \in \Sigma$, and $\sigma_i = K\pi^i \wedge \bigwedge_j L\pi^i_j$, where the π^is and π^i_js are in Π.

If $\mathcal{X}$ and $\mathcal{X}'$ are disjoint subset spaces then we write $\mathcal{X} \cup \mathcal{X}'$ for the natural union of the two; i.e., the points of $\mathcal{X} \cup \mathcal{X}'$ are $X \cup X'$ and the opens are

$$\{ U \cup V \mid U \text{ is open in } \mathcal{X},\ V \text{ is open in } \mathcal{X}' \} .$$

(We assume here that the $\emptyset$ is an open in each space, so the opens of $\mathcal{X}$ and $\mathcal{X}'$ are open in $\mathcal{X} \cup \mathcal{X}'$.) The important thing to note is that if π is bi-persistent and *both* $\mathcal{X}$ and $\mathcal{X}'$ satisfy $K\pi$, then so does $\mathcal{X} \cup \mathcal{X}'$; and if *either* $\mathcal{X}$ or $\mathcal{X}'$ satisfies $L\pi$, then so does $\mathcal{X} \cup \mathcal{X}'$.

For each i let $\phi'_i = \pi_i \wedge \Diamond K\pi^i \in \Pi$; let $\phi' = \bigvee_i \phi'_i$. Then $\vdash \phi \rightarrow \phi'$. We are done if we can show that $\vdash \phi' \rightarrow \phi$. So assume towards a contradiction that $\phi' \wedge \neg\phi$ is consistent. By completeness, there is a model $\mathcal{X}$ containing a point x such that $x, X \models \phi' \wedge \neg\phi$. Let i be such that $x, X \models \phi'_i$. Choose a neighborhood U of x such that $U \models K\pi^i$. Since ϕ_i is consistent, there is also a model $\mathcal{X}'$ of $K\pi^i \wedge \bigwedge_j L\pi^i_j$. Then in $\mathcal{X} \cup \mathcal{X}'$ we have

$$x, U \cup X' \ \models\ \pi_i \wedge K\pi^i \wedge \bigwedge_j L\pi^i_j ,$$

hence $x, U \cup X' \models \phi$. But we also assumed that ϕ is bi-persistent, and so $x, X \cup X' \models \phi$ and also $x, X \models \phi$. This is a contradiction. $\dashv$

Proposition 3.16 *There is a sentence ϕ which is bi-persistent for intersection spaces, but which is not equivalent to any $\pi \in \Pi$ on all intersection spaces.*

Proof Let $\mathcal{X}$ be the model from Example B. Our discussion there showed that the theory of intersection spaces does not have the finite model property. Recall that the point c had the property that

$$c, X \quad \models \quad \Box\Diamond\phi \ \wedge\ \Box\Diamond\neg\phi$$

for some sentence ϕ. Let $\psi \equiv \Box\Diamond\phi \wedge \Box\Diamond\neg\phi$. This ψ is bi-persistent on intersection spaces (since, as can easily be seen, any $\Box\Diamond$ sentence is bi-persistent).

Suppose towards a contradiction that ψ is equivalent to some $\pi \in \Pi$.

Let $\mathcal{Y}$ be the model obtained by adding all unions of collections of sets from $\mathcal{X}$. Then on one hand, $c, X \models_{\mathcal{X}} \pi$ because $c, X \models_{\mathcal{X}} \psi$. In addition, since $\mathcal{X}$ and $\mathcal{Y}$ agree on all sentences in Π, we see that $c, X \models_{\mathcal{Y}} \pi$. However, ψ contradicts the complete intersection axiom. So $c, X \models_{\mathcal{Y}} \neg\psi$. ⊣

Proposition 3.17 *If ϕ is bi-persistent for (finite) subset spaces, then there is some boolean combination β of atomic sentences such that $\vdash_{set\ spaces} \phi \leftrightarrow \beta$.*

Proof Let $\beta_1, \ldots, \beta_k$ be all complete boolean combinations of atomic sentences occurring in ϕ such that $\beta \wedge \phi$ is consistent under the axioms of subset spaces. Then by propositional logic

$$\vdash \ \phi \ \rightarrow \bigvee_{1 \leq i \leq n} \beta_i \ .$$

We claim that the converse also holds. If not, then by Completeness, there is a subset space $\mathcal{X} = \langle X, \mathcal{O} \rangle$, a point x, and some fixed β_i so that $x, X \models \beta_i \wedge \neg\phi$. Then since β_i is consistent with ϕ, another application of completeness gives a subset space $\mathcal{Y} = \langle Y, \mathcal{O}' \rangle$ and a point y so that $y, Y \models \beta_i \wedge \phi$.

Now form a new space $\mathcal{Z} = \langle Z, \mathcal{O}'' \rangle$ by taking Z to be the disjoint union of X and Y and identifying x and y. Call this point z. We similarly take $\mathcal{O}''$ to be the collections of unions of sets in $\mathcal{O}$ with sets in $\mathcal{O}'$, identifying x with y. The point is that it is possible to interpret the atomic sentences in $\mathcal{Z}$ because both (x, X) and (y, Y) satisfy β, and β gives complete information about the atomic sentences which appear in ϕ. (The other atomic sentences may be interpreted arbitrarily.)

Recall that we assume ϕ to be bi-persistent for subset spaces. Note that

$$z, Z \models (\Diamond\phi \ \wedge \ \Diamond\neg\phi) \ ,$$

and this contradicts the bi-persistence assumption. ⊣

3.5 The Finite Model Property

Lemma 3.18 *Let π and π' belong to Π, and suppose that $\pi \wedge K\pi'$ is consistent in topologic. Then there is a finite topological space $\mathcal{X} = \langle X, \mathcal{O} \rangle$, a point $x \in X$ and an open $u \in \mathcal{O}$ so that $x, u \models_{\mathcal{X}} \pi \wedge K\pi'$.*

Proof If $\pi \wedge K\pi'$ is consistent, then it is satisfiable. Then the sentence $\mu = \pi \wedge \Diamond K\pi'$ is also satisfiable. Now $\mu \in \Pi$, so μ_* is a satisfiable sentence of $\mathcal{L}(S4)$. By McKinsey's Theorem 3.6, let $\mathcal{X}$ be a finite topological space containing a point x so that $x \models_{McK} \mu_*$. By Proposition 3.5, $x, X \models_{top} \mu$. Thus there is some $u \in \mathcal{O}$ so that $x, u \models_{top} K\pi'$. By bi-persistence, $x, u \models_{top} \pi$ as well. ⊣

Theorem 3.19 (Georgatos [G2]) *Suppose that ϕ is a consistent sentence of topologic. Then ϕ is satisfiable in a finite topological space.*

Proof We may assume that ϕ is in prime normal form

$$\pi \wedge K\pi' \wedge L\mu_1 \wedge \cdots \wedge L\mu_n .$$

For $1 \leq i \leq n$, the sentence $\mu_i \wedge K\pi'$ is consistent. By Lemma 3.18, let $\mathcal{X}_i$ be a finite space containing a point x_i and an open u_i so that $x_i, u_i \models_{\mathcal{X}_i} \mu_i \wedge K\pi'$. Also, let $\mathcal{Y}$ be a finite space containing y, v so that $y, v \models_{\mathcal{Y}} \pi \wedge K\pi'$.

Let $\mathcal{X}$ be the disjoint union of $\mathcal{X}_1, \ldots, \mathcal{X}_n$ and $\mathcal{Y}$. $\mathcal{X}$ is a finite topological space. Let $u = u_1 \cup \cdots \cup u_n \cup v$. Using the bi-persistence of all of the sentences involved, we see that $y, u \models_{\mathcal{X}} \phi$. ⊣

4 References

[CM] Chandy, M. and Misra, J. 1986. How processes learn. *Distributed Computing* 1:40-52.

[FHV] Fagin, R., Halpern, J., and Vardi, M. 1988. A Model-Theoretic Analysis of Knowledge. Research Report RJ 6461, IBM.

[G1] Georgatos, K. Ph.D. Dissertation. City University of New York, 1993.

[G2] Georgatos, K. Knowledge theoretic properties of topological spaces. In Michael Masuch and Polos Laszlo, editors, *Knowledge Representation and Uncertainty*, Lecture Notes in Computer Science. Springer-Verlag, Berlin, New York, 1994.

[G3] Georgatos, K. Reasoning about knowledge on computation trees. Manuscript, City University of New York, March, 1994.

[Go] Goldblatt, R. 1987. *Logics of Time and Computation*, CSLI Lecture Notes No. 7.

[Hi] Hintikka, J. 1962. *Knowledge and Belief*, Ithaca: Cornell University Press.

[HM] Halpern, J. and Moses, Y. 1984. Knowledge and common knowledge in a distributed environment, *Proc. 3rd ACM Symposium on Principles of Distributed Computing*, New York: ACM, pp. 50-61.

[Ma] Makinson, D. 1966. On some completeness theorems in modal logic. *Zeit. f. Math. Logik* 12:379-384.

[Pa] Parikh, R. 1984. Logics of knowledge, games and dynamic logic, *FST-TCS, LNCS 181*. Berlin; New York: Springer-Verlag, pp. 202-222.

[PR] Parikh, R. and Ramanujam, R. 1985. Distributed computing and the logic of knowledge. *Logics of Programs, LNCS 193*, R. Parikh (ed.). Berlin; New York: Springer-Verlag, pp. 256-268.

[Sh] Shin, S. J. 1991. Valid Rreasoning and Visual Representation, PhD Thesis, Stanford University.

[Vi] Vickers, S. 1989. *Topology via Logic*, Cambridge: Cambridge University Press.