Technical Report No. 154

Some Observations on n-Valued Disjointly Separable Functions

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March 9, 1984

This is the result of work relating to C525 and C890, 1983-1984.
This paper will appear in the Proceedings of the Fourteenth International Symposium on Multiple-valued Logic, Winnipeg, May, 1984.

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ABSTRACT

A definition is given for n-valued disjointly separable functions, and examples of such functions are provided. It is shown that 2-valued disjointly separable functions coincide with linearly separable Boolean functions. Examples are provided of functions which are n-valued disjointly separable for each $n \geq 2$. This work gives a basis for future enumerations of n-valued disjointly separable functions of m arguments, for various values of n,m. Such enumerations are already known for $n=2$, $m \leq 8$.

I. INTRODUCTION

We generalize linearly separable Boolean functions (binary threshold functions) [1, 5, 9, 10] to n-valued disjointly separable functions, where $n \geq 2$. Since the definition we give for 2-valued disjointly separable functions differs from a standard definition for linearly separable Boolean functions [p.584,4], we prove that these two notions are equivalent.

While there have been other multiple-valued generalizations of linearly separable Boolean functions [6, 7], our approach is different. Our presentation of n-valued disjointly separable functions exploits disjoint operators [3] in a new way.

Examples are given of n-valued functions which for each $n \geq 2$ are n-valued disjointly separable. Attention is then concentrated on 3-valued disjointly separable functions of three arguments. Examples of such functions are given. An example is given of a 3-valued function of three arguments which is not a 3-valued disjointly separable function.

While no enumerations are given, this work lays a basis for future work in this direction. For n-valued disjointly separable functions of m arguments, such enumerations are already known for $n=2$, $m \leq 8$ [8].

II. DISCUSSION

For n-valued functions we use Post functions of order n[2], where the variables range over the linearly ordered values $[0, 1, \cdots, n-1]$. It is understood that e_i represents the constant i, $0 \leq i \leq n-1$, and $C_i(j)=n-1$ for $i=j$, with $C_i(j)=0$ for $i \neq j$, where $0 \leq i,j \leq n-1$. Over these linearly ordered values, $i \vee j$ represents $\max(i,j)$ and $i \wedge j$ represents $\min(i,j)$. The usual arithmetic operations of addition and multiplication are denoted by $a + b$ and ab, respectively.

Definition: Let f be an n-valued function of m arguments $x_1, \cdots, x_m$. Then f is n-valued disjointly separable if there exist thresholds T_k, weights N_{pi} and a function $S = \sum_{p=1}^{m} \sum_{i=0}^{n-1} N_{pi} C_i(x_p)$ such that $S \geq T_{n-1}$ whenever $f=n-1$; $T_{k+1} > S \geq T_k$ whenever $f=k$, $k=1,2,\cdots,n-2$; and $T_1 > S$ whenever $f=0$.

With regard to this definition, a referee of this paper has noted that the $C_i(x), i=0,1,\cdots,n-1$, may be regarded as arithmetic polynomials in x of degree n-1. Thus S involves for each of the m arguments $x_p, p=1,2,\cdots,m$, n polynomials of degree n-1.

This definition gives a generalization of linearly separable Boolean functions. Consider this definition for the case n=2. The function S becomes $S = \sum_{p=1}^{m} (N_{p0}C_0(x_p) + N_{p1}C_1(x_p))$ with $S \geq T_1$ whenever f=1, $T_1 > S$ whenever f=0. For n=2, $C_0(x_p) = \overline{x_p}$, $C_1(x_p) = x_p$. By the definition of a linearly separable Boolean function at least one of N_{p0}, N_{p1} for each $p=1,\cdots,m$ must be 0. Hence any linearly separable Boolean function is a 2-valued disjointly separable function. For the converse, consider $S' = \sum_{p=1}^{m} N'_p C_{j(p)}(x_p)$, where for each p, j(p) = 0 or 1, and the threshold T'_1 is such that $S' \geq T'_1$ whenever f=1, $T'_1 > S'$ whenever

$f=0$. Let $N'_p = \left| N_{p0} - N_{p1} \right|$; if $N_{p0}<N_{p1}$, $j(p)=1$ and if $N_{p1}<N_{p0}$, $j(p)=0$. Also $T'_1 = T_1 - \sum_{p=1}^{m} \min(N_{p0},N_{p1})$. It suffices to prove that $S' = S - \sum_{p=1}^{m} \min(N_{p0},N_{p1})$. For any p, $N_{p0}\overline{x_p} + N_{p1}x_p - N'_p C_{j(p)}(x_p) = \min(N_{p0},N_{p1})$ for $x_p=0$ and for $x_p=1$ over the cases $N_{p0}<N_{p1}$, $N_{p0}=N_{p1}$, $N_{p1}<N_{p0}$. The result follows by summation over p.

<u>Theorem 1</u>

Let $f(x_1,\cdots,x_m)$ be a 2-valued disjointly separable function, with threshold T and $S = \sum_{p=1}^{m} N_p C_{j(p)}(x_p)$, where for each p, $j(p) = 0$ or 1. We create, for each n, an n-valued disjointly separable function $g(y_1,\cdots,y_m)$, obtained from f by the substitutions $\overline{x_i} = C_0(y_i)$, $x_i = C_{n-1}(y_i)$ for each $i=1,\cdots,m$. For each n, g is n-valued disjointly separable with $S' = \sum_{p=1}^{m} N_p C_{j'(p)}(y_p)$, where for each p, $j'(p)=(n-1)j(p)$, and with n-1 thresholds satisfying $T'_1<T'_2<\cdots<T'_{n-2}<T'_{n-1}=(n-1)T$.

<u>Proof:</u> There exists an $\epsilon>0$ such that $S<T-\overline{(n-2)}\epsilon$. Set $T'_j=(n-1)(T-(n-1-j)\epsilon)$. Thus $(n-1)T=T'_{n-1}>T'_{n-2}>\cdots>T'_1$.

The weights N_p, $p=1,\cdots,m$ are taken to be positive [p.585,4]. The functions f and g are each expressible as a disjunction of conjunctive terms.

If $g=n-1$, then there is at least one conjunctive term of g which is n-1. This occurs for certain variables y_i of this term assuming 0 or n-1 values. There is a corresponding conjunctive term of f which is 1 for corresponding x_i assuming 0 or 1 values. These x_i values yield $\sum_{j\in J} N_j \geq T$, where the index set J depends on these x_i values. It follows that $\sum_{j\in J}(n-1)N_j \geq (n-1)T = T'_{n-1}$.

If $g=0$, let each y_i assume value $v(i)$. Since f is unate [Theorem 16.13, 4], at most one of $C_0(y_i)$, $C_{n-1}(y_i)$ occurs in g [p. 579,4]. If $v(i)$ is 0 or n-1, set $v'(i)=v(i)$. Otherwise if $C_0(y_i)$ appears set $v'(i) = n-1$; if $C_{n-1}(y_i)$ appears set $v'(i)=0$. Consequently $g(v(1),\cdots,v(m)) = g(v'(1),\cdots,v'(m))$. Since the $v'(i)$ values are all 0 or n-1, there is a corresponding sequence of 0,1 values for x_i which makes $f=0$. This yields $\sum_{k\in K} N_k<T-(n-2)\epsilon$ for some index set K. Using values $v(i)$, since S' contains only terms $C_j(x_p)$ where $j=0$ or n-1, there is a subset K' of K such that $\sum_{k\in K'} N_k \leq \sum_{k\in K} N_k < T-(n-2)\epsilon$. Finally $\sum_{k\in K'} (n-1)N_k<(n-1)(T-(n-2)\epsilon)=T'_1$

<u>Example 1</u>

The Boolean function $f = (x_1\wedge\overline{x_2})\vee x_3$ is 2-valued disjointly separable with $S=x_1 + \overline{x_2} + 2x_3$ and $T=2$. By Theorem 1 using $n=5$, $g=(C_4(y_1)\wedge C_0(y_2))\vee C_4(y_3)$ is 5-valued disjointly separable with $S'=C_4(y_1)+C_0(y_2)+2C_4(y_3)$. Using $\epsilon=0.01$, we have $T'_4=4(2)=8, T'_3=4(1.99)=7.96$, $T'_2=4(1.98)=7.92$, and $T'_1=4(1.97)=7.88$. If $g=4, S'\geq T'_4=8$; if $g=0, S'<T'_1=7.88$.

<u>Theorem 2</u>

The n-valued function $f = x_1\vee x_2$ is n-valued disjointly separable for each $n\geq 2$, with $S=\sum_{p=1}^{2}\sum_{i=0}^{n-1} N_{pi}C_i(x_p)$ where $N_{1i}=N_{2i}=(2^i-1)/(n-1)$ for each $i=0,\cdots,n-1$ and $T_k=2^k-1$ for $k=1,\cdots,n-1$.

<u>Proof</u>: Let $f=k$, $0\leq k\leq n-1$. The maximum value of S is $(2^k-1)+(2^k-1)=2^{k+1}-2$. The minimum value of S is 2^k-1. Thus for $T_k=2^k-1$, $k=1,\cdots,n-1$, $T_{k+1}>S$ and $S\geq T_k$. This completes the proof.

While the total number of possible n-valued functions of m arguments is large, n^{n^m}, the requirements $T_{k+1}>S\geq T_k$, $k=1,\cdots,n-2$ are stringent. To study this stringency, we investigate the 3-valued case for a number of examples, using three arguments. In each example we first determine the maximum value of T_2 and the minimum value of T_1, then establish that the function is 3-valued disjointly separable by showing that $T_2>S\geq T_1$.

Example 2

The 3-valued function $f_2=C_1(x_1)\vee(x_2\wedge x_3)$ is 3-valued disjointly separable, with $S=6C_1(x_1) + 3C_2(x_2) + 2C_1(x_2) + 3C_2(x_3) + 2C_1(x_3)$ and $T_1=7$, $T_2=12$.

When $f_2=2$: $x_1=1$ or $x_2=x_3=2$; so $\max(T_2)=12$.
When $f_2=0$: ($x_1=0$ or 2) and ($x_2=0$ or $x_3=0$); so $\min(T_1)>6$.
When $f_2=1$: ($x_1=0$ or 2) and $\min(x_2,x_3)=1$; so $8\leq S\leq 10$.

Example 3

The 3-valued function $f_3=(e_1\wedge(C_1(x_2)\vee C_2(x_3)))\vee x_1$ is 3-valued disjointly separable, with $S=4C_2(x_1) + C_1(x_1) + C_1(x_2) + C_2(x_3)$ and $T_1=1$, $T_2=8$.

When $f_3=2$: $x_1=2$; so $\max(T_2)=8$.
When $f_3=0$: $x_1=0$ and ($x_2=0$ or 2) and ($x_3=0$ or 1); so $\min(T_1)>0$.
When $f_3=1$: ($x_1=0$ and ($x_2=1$ or $x_3=2$)) or $x_1=1$; so $2\leq S\leq 6$.

Example 4

The 3-valued function $f_4=C_2(x_1)\wedge(x_2 \vee x_3)$ is 3-valued disjointly separable, with $S=6C_2(x_1) + 3C_2(x_2) + C_1(x_2) + 3C_2(x_3) + C_1(x_3)$ and $T_1=13$, $T_2=18$.

When $f_4=2$: $x_1=2$ and ($x_2=2$ or $x_3=2$); so $\max(T_2)=18$.
When $f_4=0$: ($x_1=0$ or 1) or $x_2=x_3=0$; so $\min(T_1)>12$.
When $f_4=1$: $x_1=2$ and $\max(x_2, x_3)=1$; so $14\leq S\leq 16$.

For the general case, the weights and threshold can be determined by linear programming. If the n-valued function of m arguments is not n-valued disjointly separable, the linear programming will fail.

We give one further result for n=3, m=3, an example of a 3-valued function which is not 3-valued disjointly separable.

Theorem 3

The 3-valued function $h= w \vee (x\wedge y)$ is not a 3-valued disjointly separable function.

Proof: For ease of understanding, we rewrite the coefficients N_{pi} appearing in S as follows:

$$S = aC_0(w) + fC_1(w) + kC_2(w) + d_1C_0(x) + g_1C_1(x) + b_1C_2(x) + d_2C_0(y) + g_2C_1(y) + b_2C_2(y).$$

We attempt to maximize T_2 and minimize T_1 such that $T_1<T_2$ and whenever h=2, $T_2\leq S$; whenever h=0, $S<T_1$; whenever h=1, $T_1\leq S<T_2$.

Consider h=2. This occurs for w=2 or x=y=2. It follows that $T_2/2=\min(k+d_1+d_2, k+d_1+g_2, k+d_1+b_2, k+g_1+d_2, k+g_1+g_2, k+g_1+b_2, k+b_1+d_2, k+b_1+g_2, k+b_1+b_2, a+b_1+b_2, f+b_1+b_2)$. This yields eleven possible values for T_2.

Consider h=0. This occurs for w=0 and (x=0 or y=0). It follows that $T_1/2=\max(a+d_1+d_2, a+d_1+g_2, a+d_1+b_2, a+g_1+d_2, a+b_1+d_2)+\epsilon_1$, where $\epsilon_1>0$. This yields five possible values for T_1.

Thus there are 11 x 5=55 cases to consider in this theorem. We have found proofs for all these cases, and give below two representative proofs. These two proofs establish 12 of the 55 cases. The other 43 cases are established through proofs in a similar vein.

The following table shows the constraints which arise when h=1.

h=1 when			corresponding constraints
w	x	y	
0	1	1	$T_1/2 \leq a+g_1+g_2 < T_2/2$
0	1	2	$T_1/2 \leq a+g_1+b_2 < T_2/2$
0	2	1	$T_1/2 \leq a+b_1+g_2 < T_2/2$
1	0	0	$T_1/2 \leq f+d_1+d_2 < T_2/2$
1	0	1	$T_1/2 \leq f+d_1+g_2 < T_2/2$
1	1	0	$T_1/2 \leq f+g_1+d_2 < T_2/2$
1	1	1	$T_1/2 \leq f+g_1+g_2 < T_2/2$
1	1	2	$T_1/2 \leq f+g_1+b_2 < T_2/2$
1	2	1	$T_1/2 \leq f+b_1+g_2 < T_2/2$

1. Consider $T_1/2=a+d_1+d_2+\epsilon_1$, $\epsilon_1>0$. For this proof $T_2/2$ may take any of its eleven possible values.

By definition of T_1, $T_1/2=a+d_1+d_2+\epsilon_1\geq a+d_1+g_2+\epsilon_1$. Hence $d_2\geq g_2$. Likewise $T_1/2=a+d_1+d_2+\epsilon_1\geq a+g_1+d_2+\epsilon_1$ and $d_1\geq g_1$. Thus $d_1+d_2\geq g_1+g_2$. We will obtain a contradiction to this.

From the table for w=0,x=1,y=1, we have $a+g_1+g_2\geq T_1/2$, that is, $a+g_1+g_2\geq a+d_1+d_2+\epsilon_1,\epsilon_1>0$. Hence $g_1+g_2>d_1+d_2$. This completes the proof by contradiction, and establishes 11 of the 55 cases.

2. Consider:

$$T_2/2 = k+d_1+d_2$$
$$T_1/2 = a+d_1+b_2+\epsilon_1,\epsilon_1>0.$$

This is one of the 55 cases.

From the table for w=0,x=1,y=2, $a+g_1+b_2\geq T_1/2$, that is, $a+g_1+b_2\geq a+d_1+b_2+\epsilon_1$. Hence $g_1>d_1$. We will obtain a contradiction to this.

From the table for w=1, x=1, y=2, $f+g_1+b_2<T_2/2$. By definition of T_2, $T_2/2=k+d_1+d_2\leq a+b_1+b_2$. Therefore $f+g_1<a+b_1$. That is, $f<a+b_1-g_1$.

From the table for w=1, x=0, y=0, $f+d_1+d_2\geq T_1/2$. By definition of T_1, $T_1/2=a+d_1+b_2+\epsilon_1\geq a+b_1+d_2+\epsilon_1$. Therefore $f+d_1+d_2\geq a+b_1+d_2+\epsilon_1$. Since $f<a+b_1-g_1$ we have $a+b_1-g_1+d_1+d_2>a+b_1+d_2+\epsilon_1$. That is, $d_1>g_1$. This completes the proof by contradiction for this case.

III. SUMMARY

We have presented a generalization of linearly separable Boolean functions to n-valued disjointly separable functions. We have given a number of examples of such functions, and have shown a 3-valued function which is not a 3-valued disjointly separable function.

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