Geometrical structures and modal logic

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Abstract

Although, in natural language, space modalities are used as frequently as time modalities, the logic of time is a well-established branch of modal logic whereas the same cannot be said of the logic of space. The reason is probably in the more simple mathematical structure of time: a set of moments together with a relation of precedence. Such a relational structure is suited to a modal treatment. The structure of space is more complex: several sorts of geometrical beings as points and lines together with binary relations as incidence or orthogonality, or only one sort of geometrical beings as points but ternary relations as collinearity or betweeness. In this paper, we define a general framework for axiomatizing modal logics which Kripke semantics is based on geometrical structures: structures of collinearity, projective structures, orthogonal structures.

1 Introduction

Although, in natural language, space modalities are used as frequently as time modalities, the logic of time is a well-established branch of modal logic [5] whereas the same cannot be said of the logic of space. The reason is probably in the more simple mathematical structure of time: a set of moments together with a relation of precedence. Such a relational structure is suited to a modal treatment. The structure of space is more complex: several sorts of geometrical beings as points and lines together

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with binary relations as incidence [7] [8] or orthogonality [6], or only one sort of geometrical beings as points but ternary relations as collinearity or betweeness [11]. Such relational structures are not suited to a modal treatment.

Structures of collinearity, first example, consist of a set of points together with a ternary relation of collinearity between points. They cannot constitute the standard semantics of the modal logic of collinearity. To overcome this problem, a frame of collinearity is associated to every structure of collinearity in the following way. Let $\underline{S} = (P, C)$ be a structure of collinearity, with P the set of points and C the ternary relation of collinearity between points. Let W be the graph of the relation C. For every $i, j \in \{1, 2, 3\}$, let $\equiv_{ij}$ be the binary relation on W defined by $(X_1, X_2, X_3) \equiv_{ij} (Y_1, Y_2, Y_3)$ iff $X_i = Y_j$. Intuitively, each element (X, Y, Z) of W can be considered either as X, Y or Z. Let $u = (X_1, X_2, X_3)$, $v = (Y_1, Y_2, Y_3)$ and $w = (Z_1, Z_2, Z_3)$. The expression $(\exists x \in W)(u \equiv_{i1} x \wedge v \equiv_{j2} x \wedge w \equiv_{k3} x)$ is equivalent to $C(X_i, Y_j, Z_k)$. Therefore, $W(\underline{S}) = (W, \equiv_{ij})$ contains in some sense the whole information of $\underline{S}$. $W(\underline{S})$ is the frame of collinearity over $\underline{S}$. It satisfies the properties detailed in section 4.2. A frame of collinearity is a relational structure of the form $(W, \equiv_{ij})$ that satisfies the same properties. It can be proved that any frame of collinearity is isomorphic to a frame of collinearity over some structure of collinearity and that the set of all structures of collinearity is categorically equivalent to the set of all frames of collinearity. Therefore, frames of collinearity can constitute the standard semantics of the modal logic of collinearity.

Projective structures, second example, consist of a set of points and a set of lines together with a binary relation of incidence between points and lines. They cannot constitute the standard semantics of the modal logic of projective geometry. To overcome this problem, a projective frame is associated to every projective structure in the following way. Let $\underline{S} = (P, L, in)$ be a projective structure, with P the set of points, L the set of lines and in the binary relation of incidence between points and lines. Let W be the graph of the relation in. Let $\equiv_{11}$ and $\equiv_{22}$ be the binary relations on W defined by $(X, x) \equiv_{11} (Y, y)$ iff $X = Y$ and $(X, x) \equiv_{22} (Y, y)$ iff $x = y$. Intuitively, each element (X, x) of W can be considered either as X or x. Let $u = (X, x)$ and $v = (Y, y)$. The expression $u \equiv_{11} \circ \equiv_{22} v$ is equivalent to $X \; in \; y$. Therefore, $W(\underline{S}) = (W, \equiv_{11}, \equiv_{22})$ contains in some sense the whole information of $\underline{S}$. $W(\underline{S})$ is the projective frame over $\underline{S}$. It satisfies the properties detailed in section 7.1.2. A projective frame is a relational structure of the form $(W, \equiv_{11}, \equiv_{22})$ that satisfies the same properties. It can be proved that any projective frame is isomorphic to a projective frame over some projective structure and that the set of all projective structures is categorically equivalent to the set of all projective frames. Therefore, projective frames can constitute the standard semantics of the modal logic of projective geometry.

Section 2 introduces point n-frames and n-arrow frames and gives the proof of their categorial equivalence. Section 3 extends Vakarelov's basic arrow logic [12] [13]. Its standard semantics is the set of all n-arrow frames. Section 4 describes an example of point 3-frames: the structures of collinearity, proves its categorial equivalence with the associated example of 3-arrow frames: frames of collinearity, and identifies the modal logic with standard semantics in the set of all frames of collinearity. Sections 5 and 6 extend the results of sections 2 and 3 to sorted point n-frames and sorted n-arrow frames. Section 7.1 describes an example of sorted point 2-frames: the projective structures, proves its categorial equivalence with the associated example of sorted 2-arrow frames: projective frames, and identifies the modal logic with standard semantics in the set of all projective frames. Section 7.2 describes an example of sorted point 3-frames: the orthogonal structures, proves its categorial equivalence with the associated example of sorted 3-arrow frames: orthogonal frames, and identifies the modal logic with standard semantics in the set of all orthogonal frames.

2 Point n-frames and n-arrow frames

Let $n \geq 2$ and $(n) = \{1, \ldots, n\}$. This section is devoted to the proof of the categorial equivalence between point n-frames and n-arrow frames.

2.1 Point n-frames

A *point n-frame* consists of a non-empty set S together with a n-ary relation R on S such that, for every $X \in S$, there exists $X_1, \ldots, X_n \in S$ and there exists $i \in (n)$ such that $R(X_1, \ldots, X_n)$ and $X_i = X$. The class of all point n-frames is denoted by Σ_n and is considered as a category with morphisms the usual homomorphisms between relational structures. Namely, let $\underline{S} = (S, R)$ and $\underline{S}' = (S', R')$ be point n-frames. Then f is called a *homomorphism* from $\underline{S}$ into $\underline{S}'$ if, for every $X_1, \ldots, X_n \in S$, $R(X_1, \ldots, X_n)$ only if $R'(f(X_1), \ldots, f(X_n))$. A one-to-one f is called an *isomorphism* from $\underline{S}$ into $\underline{S}'$ if, for every $X_1, \ldots, X_n \in S$, $R'(f(X_1), \ldots, f(X_n))$ only if $R(X_1, \ldots, X_n)$.

2.2 n-arrow frames

A *n-arrow frame* consists of a non-empty set W of *tips* together with n^2 binary relations $\equiv_{ij}$ on W such that, for every $i, j, k \in (n)$:

- for every $u \in W$, $u \equiv_{ii} u$,
- for every $u, v \in W$, $u \equiv_{ij} v$ only if $v \equiv_{ji} u$,
- for every $u, v, w \in W$, $u \equiv_{ij} v$ and $v \equiv_{jk} w$ only if $u \equiv_{ik} w$.

The n-arrow frame $\underline{W} = (W, \equiv_{ij})$ is *normal* if:

- for every $u, v \in W$, $u \equiv_{ii} v$, for every $i \in (n)$, only if $u = v$.

The class of all normal n-arrow frames is denoted by Φ_n and is considered as a category with morphisms the usual homomorphisms between relational structures.

2.3 From point n-frames to n-arrow frames

Let $\underline{S} = (S, R)$ be a point n-frame. Let $W = \{(X_1, \ldots, X_n)\colon X_1, \ldots, X_n \in S$ and $R(X_1, \ldots, X_n)\}$. For every $i, j \in (n)$, let $\equiv_{ij}$ be the binary relation on W defined by $(X_1, \ldots, X_n) \equiv_{ij} (Y_1, \ldots, Y_n)$ iff $X_i = Y_j$.

Lemma 1 *$W(\underline{S}) = (W, \equiv_{ij})$ is a normal n-arrow frame.*

Example 1 *Suppose $n = 3$. It can be proved that if, for every $X, Y \in S$, $R(X, Y, X)$ then, for every $i, j \in (3)$ and for every $x, y \in W$, there exists $u \in W$ such that $x \equiv_{i1} u$, $y \equiv_{j2} u$ and $x \equiv_{i3} u$.*

2.4 From n-arrow frames to point n-frames

Let $\underline{W} = (W, \equiv_{ij})$ be a normal n-arrow frame. For every $\alpha_1, \ldots, \alpha_n \in \mathcal{P}(W)$, if:

- for every $i, j \in (n)$ and for every $u, v \in W$, $u \in \alpha_i$ and $v \in \alpha_j$ only if $u \equiv_{ij} v$,
- for every $i, j \in (n)$ and for every $u, v \in W$, $u \in \alpha_i$ and $u \equiv_{ij} v$ only if $v \in \alpha_j$,
- $\alpha_1 \cup \ldots \cup \alpha_n \neq \emptyset$

then $(\alpha_1, \ldots, \alpha_n)$ is a *generalized point* of $\underline{W}$. For every $u \in W$ and for every $i \in (n)$, let $i(u) = (\equiv_{i1} (u), \ldots, \equiv_{in} (u))$. Direct calculations would lead to the conclusion that:

Lemma 2 *For every $u, v \in W$ and for every $i, j \in (n)$, $i(u) = j(v)$ iff $u \equiv_{ij} v$.*

Lemma 3 *For every generalized point $(\alpha_1, \ldots, \alpha_n)$ of $\underline{W}$, there exists $u \in W$ and $i \in (n)$ such that $i(u) = (\alpha_1, \ldots, \alpha_n)$.*

Let S be the set of the generalized points of $\underline{W}$. Let R be the n-ary relation on S defined by $R(X_1, \ldots, X_n)$ iff there exists $u \in W$ such that, for every $i \in (n)$, $X_i = i(u)$.

Lemma 4 *$S(\underline{W}) = (S, R)$ is a point n-frame.*

Example 2 *Suppose $n = 3$. It can be proved that if, for every $i, j \in (3)$ and for every $x, y \in W$, there exists $u \in W$ such that $x \equiv_{i1} u$, $y \equiv_{j2} u$ and $x \equiv_{i3} u$ then, for every $X, Y \in S$, $R(X, Y, X)$.*

2.5 Representation theorems

Let $\underline{W} = (W, \equiv_{ij})$ be a normal n-arrow frame and $\underline{W}' = W(S(\underline{W}))$. For every $u \in W$, let $g(u) = (1(u), \ldots, n(u))$. Direct calculations would lead to the conclusion that g is an isomorphism from $\underline{W}$ into $\underline{W}'$. Therefore:

Lemma 5 *$\underline{W}$ and $\underline{W}'$ are isomorphic.*

Let $\underline{S} = (S, R)$ be a point n-frame and $\underline{S}' = S(W(\underline{S}))$. For every $X \in S$, let $f(X) = (1(X), \ldots, n(X))$ where $i(X) = \{(X_1, \ldots, X_n)\colon X_1, \ldots, X_n \in S$, $R(X_1, \ldots, X_n)$ and $X_i = X\}$. Direct calculations would lead to the conclusion that f is an isomorphism from $\underline{S}$ into $\underline{S}'$. Consequently:

Lemma 6 *$\underline{S}$ and $\underline{S}'$ are isomorphic.*

Direct calculations would lead to the conclusion that the mapping S: $\underline{W} \rightarrow S(\underline{W})$ is a functor from Φ_n into Σ_n and that the mapping W: $\underline{S} \rightarrow W(\underline{S})$ is a functor from Σ_n into Φ_n. Therefore:

Theorem 1 *The categories Σ_n and Φ_n are equivalent.*

3 Basic arrow logic

Let $n \geq 2$. This section introduces a modal logic with standard semantics in the class of all normal n-arrow frames.

3.1 Language

The linguistic basis of basic arrow logic is the propositional calculus. Let VAR be the set of its *atomic formulas*. For every $i, j \in (n)$, the modal operator $[\equiv_{ij}]$ is added to the standard propositional formalism and, for every $i \in (n)$, the modal operator $[\not\equiv_{ii}]$ is added to the standard propositional formalism. Let $[\neq]A = [\not\equiv_{11}]A \wedge \ldots \wedge [\not\equiv_{nn}]A$ and $[U]A = A \wedge [\neq]A$.

3.2 Semantics

A *general n-arrow frame* consists of a non-empty set W together with, for every $i, j \in (n)$, a binary relation $\equiv_{ij}$ on W such that $(W, \equiv_{ij})$ is a n-arrow frame and, for every $i \in (n)$, a binary relation $\not\equiv_{ii}$ on W. A general n-arrow frame $\underline{W} = (W, \equiv_{ij}, \not\equiv_{ii})$ is *$\neq$-standard* if, for every $u, v \in W$, $u \neq v$ iff there exists $i \in (n)$ such that $u \not\equiv_{ii} v$. A $\neq$-standard frame $\underline{W} = (W, \equiv_{ij}, \not\equiv_{ii})$ is *quasi-standard* if, for every $i \in (n)$, $\not\equiv_{ii}$ is the complement of $\equiv_{ii}$.

Lemma 7 *If the general n-arrow frame $\underline{W} = (W, \equiv_{ij}, \not\equiv_{ii})$ is quasi-standard then the n-arrow frame $(W, \equiv_{ij})$ is normal.*

Let $\underline{W} = (W, \equiv_{ij}, \not\equiv_{ii})$ be a general n-arrow frame. A *valuation* on $\underline{W}$ is a mapping which assigns a subset of W to every atomic formula. A *($\neq$-standard, quasi-standard) general n-arrow model* is a structure of the form $\mathcal{M} = (W, \equiv_{ij}, \not\equiv_{ii}, m)$ where $\underline{W} = (W, \equiv_{ij}, \not\equiv_{ii})$ is a ($\neq$-standard, quasi-standard) general n-arrow frame and m is a valuation on $\underline{W}$. The satisfiability relation in $\mathcal{M}$ between a formula A and a possible world $u \in W$ is defined in the following way:

- $u \models_{\mathcal{M}} A$ iff $u \in m(A)$, A atomic formula,
- $u \models_{\mathcal{M}} \neg A$ iff $u \not\models_{\mathcal{M}} A$,
- $u \models_{\mathcal{M}} A \wedge B$ iff $u \models_{\mathcal{M}} A$ and $u \models_{\mathcal{M}} B$,
- for every $i, j \in (n)$, $u \models_{\mathcal{M}} [\equiv_{ij}]A$ iff, for every possible world $v \in W$, $u \equiv_{ij} v$ only if $v \models_{\mathcal{M}} A$,
- for every $i \in (n)$, $u \models_{\mathcal{M}} [\not\equiv_{ii}]A$ iff, for every possible world $v \in W$, $u \not\equiv_{ii} v$ only if $v \models_{\mathcal{M}} A$.

If $\mathcal{M}$ is $\neq$-standard then:

Lemma 8 *For every formula A and for every possible world $u \in W$:*

- $u \models_{\mathcal{M}} [\neq]A$ *iff, for every possible world $v \in W$, $u \neq v$ only if $v \models_{\mathcal{M}} A$,*
- $u \models_{\mathcal{M}} [U]A$ *iff, for every possible world $v \in W$, $v \models_{\mathcal{M}} A$.*

A formula is *valid* in a general n-arrow model when it is satisfied in every possible world of this model. A schema is *valid* in a general n-arrow frame if every instance of the schema is valid in every model on this frame. A schema is *valid* in a class of general n-arrow frames if it is valid in every frame of this class. Let Σ and Σ' be two classes of general n-arrow frames. Σ is *modally definable* in Σ' by a schema A if, for every frame $\underline{W} \in \Sigma'$, A is valid in $\underline{W}$ iff $\underline{W} \in \Sigma$.

Lemma 9 *The quasi-standard n-arrow frames are modally definable in the class of all $\neq$-standard n-arrow frames by the conjunction of the following schemata:*

- *for every $i \in (n)$, $[\equiv_{ii}]A \wedge [\not\equiv_{ii}]A \rightarrow [U]A$,*
- *for every $i \in (n)$, $<\equiv_{ii}> [\neq]A \rightarrow [\not\equiv_{ii}]A$.*

3.3 Axiomatics

Together with the classical tautologies, all the instances of the following schemata are axioms of BAL_n:

- for every $i, j \in (n)$, $[\equiv_{ij}](A \rightarrow B) \rightarrow ([\equiv_{ij}]A \rightarrow [\equiv_{ij}]B)$,
- for every $i \in (n)$, $[\not\equiv_{ii}](A \rightarrow B) \rightarrow ([\not\equiv_{ii}]A \rightarrow [\not\equiv_{ii}]B)$,
- $A \rightarrow [\neq] <\neq> A$,
- $[U]A \rightarrow [\neq][\neq]A$,
- for every $i, j \in (n)$, $[U]A \rightarrow [\equiv_{ij}]A$,
- for every $i \in (n)$, $[\equiv_{ii}]A \wedge [\not\equiv_{ii}]A \rightarrow [U]A$,
- for every $i \in (n)$, $<\equiv_{ii}> [\neq]A \rightarrow [\not\equiv_{ii}]A$,
- for every $i \in (n)$, $[\equiv_{ii}]A \rightarrow A$,
- for every $i, j \in (n)$, $A \rightarrow [\equiv_{ij}] <\equiv_{ji}> A$,
- for every $i, j, k \in (n)$, $[\equiv_{ik}]A \rightarrow [\equiv_{ij}][\equiv_{jk}]A$.

Together with the *modus ponens*, the following schemata are inference rules of BAL_n:

- for every $i, j \in (n)$, the $[\equiv_{ij}]$-necessitation rule is: if $\vdash_{BAL_n} A$ then $\vdash_{BAL_n} [\equiv_{ij}]A$,
- for every $i \in (n)$, the $[\not\equiv_{ii}]$-necessitation rule is: if $\vdash_{BAL_n} A$ then $\vdash_{BAL_n} [\not\equiv_{ii}]A$,
- the irreflexivity rule is: if B is an atomic formula not in A and $\vdash_{BAL_n} ([\neq]B \rightarrow B) \vee A$ then $\vdash_{BAL_n} A$.

Theorem 2 *The theorems of BAL_n are valid in every quasi-standard n-arrow model.*

The $\neq$-standard n-arrow frames are not modally definable in the class of all general n-arrow frames. Nevertheless, $\neq$-standard n-arrow frames can be characterized by an inference rule: the irreflexivity rule. The irreflexivity rule have been introduced by Gabbay [4] and studied by de Rijke [10]. Usually, the axiomatization of modal logic does not contain any rule of this kind [9]. If the accessibility relation associated to a modal operator is required to be irreflexive then the irreflexivity rule makes the completeness proof easier.

3.4 Completeness

This section is devoted to the proof of the completeness of BAL_n for the class of all quasi-standard n-arrow models. A formula A is *consistent* when $\not\vdash_{BAL_n} \neg A$. A finite set $\{A_1, \ldots, A_n\}$ of formulas is *consistent* when the formula $A_1 \wedge \ldots \wedge A_n$ is consistent. An infinite set of formulas is *consistent* when every of its finite subset is consistent. A set of formulas is *maximal* when, for every formula A, either A or $\neg A$ belongs to the set. A set of formulas is a *$\neq$-theory* if there exists an atomic formula B such that the formula $\neg([\neq]B \rightarrow B)$ belongs to the set. Complicated calculations would lead to the conclusion that every consistent set of formulas is a subset of a maximal consistent $\neq$-theory (see [2] for details). Let W be the set of all the maximal consistent $\neq$-theories. For every $i, j \in (n)$, let $\equiv_{ij}$ be the binary relation on W defined by $\Gamma \equiv_{ij} \Delta$ iff $\{A\colon [\equiv_{ij}]A \in \Gamma\} \subseteq \Delta$. Direct calculations would lead to the conclusion that, for every $i, j, k \in (n)$:

- for every $\Gamma \in W$, $\Gamma \equiv_{ii} \Gamma$,
- for every $\Gamma, \Delta \in W$, $\Gamma \equiv_{ij} \Delta$ only if $\Delta \equiv_{ji} \Gamma$,
- for every $\Gamma, \Delta, \Phi \in W$, $\Gamma \equiv_{ij} \Delta$ and $\Delta \equiv_{jk} \Phi$ only if $\Gamma \equiv_{ik} \Phi$.

Consequently, $(W, \equiv_{ij})$ is a n-arrow frame. For every $i \in (n)$, let $\not\equiv_{ii}$ be the binary relation on W defined by $\Gamma \not\equiv_{ii} \Delta$ iff $\{A\colon [\not\equiv_{ii}]A \in \Gamma\} \subseteq \Delta$. The following three lemmas imply that, for every $\Gamma, \Delta, \Phi \in W$ and for every sequence $(R_1, \ldots, R_k), (S_1, \ldots, S_l)$ of elements of $\{\equiv_{ij}, \not\equiv_{ii}\}$, if $\Gamma R_1 \circ \ldots \circ R_k \Delta$ and $\Gamma S_1 \circ \ldots \circ S_l \Phi$ then either $\Delta = \Phi$ or there exists $m \in (n)$ such that $\Delta \not\equiv_{mm} \Phi$.

Lemma 10 *For every $\Gamma, \Delta, \Phi \in W$ and for every $i, j \in (n)$, if $\Gamma \not\equiv_{ii} \Delta \not\equiv_{jj} \Phi$ then either $\Gamma = \Phi$ or there exists $k \in (n)$ such that $\Gamma \not\equiv_{kk} \Phi$.*

Lemma 11 *For every $\Gamma, \Delta \in W$ and for every $i, j \in (n)$, if $\Gamma \equiv_{ij} \Delta$ then either $\Gamma = \Delta$ or there exists $k \in (n)$ such that $\Gamma \not\equiv_{kk} \Delta$.*

Lemma 12 *For every $\Gamma, \Delta \in W$ and for every $i \in (n)$, if $\Gamma \not\equiv_{ii} \Delta$ then there exists $j \in (n)$ such that $\Delta \not\equiv_{jj} \Gamma$.*

Let $\underline{W} = (W, \equiv_{ij}, \not\equiv_{ii})$. Let m be the valuation on $\underline{W}$ defined by $m(A) = \{\Gamma\colon \Gamma \in W \text{ and } A \in \Gamma\}$, A atomic formula. Let $\mathcal{M} = (\underline{W}, m)$. Direct calculations would lead to the conclusion that, for every $\Gamma \in W$ and for every formula A, $\Gamma \models_{\mathcal{M}} A$ iff $A \in \Gamma$. $\mathcal{M}$ is the *canonical model* of BAL_n. Let A be a consistent formula. There exists $\Gamma \in W$ such that $A \in \Gamma$ and $\Gamma \models_{\mathcal{M}} A$. Let $W^\circ = \{\Delta\colon\ \Delta \in W$ and there exists a sequence $(R_1, \ldots, R_k)$ of elements of $\{\equiv_{ij}, \not\equiv_{ii}\}$ such that $\Gamma R_1 \circ \ldots \circ R_k \Delta\}$. Let $\equiv^\circ_{ij}, \not\equiv^\circ_{ii}$ be the restrictions of $\equiv_{ij}, \not\equiv_{ii}$ to W°. Direct calculations would lead to the conclusion that $\underline{W}^\circ = (W^\circ, \equiv^\circ_{ij}, \not\equiv^\circ_{ii})$ is a general n-arrow frame. Let m° be the restriction of m to W°. Let $\mathcal{M}^\circ = (\underline{W}^\circ, m^\circ)$. Direct calculations would lead to the conclusion that $\Gamma \models_{\mathcal{M}^\circ} A$. Moreover, for every $\Delta, \Phi \in W^\circ$, either $\Delta = \Phi$ or there exists $i \in (n)$ such that $\Delta \not\equiv_{ii} \Phi$. Let $\Delta \in W^\circ$. Since Δ is a $\neq$-theory, then there exists an atomic formula B such that $\neg([\neq]B \rightarrow B) \in \Delta$. Therefore, $[\neq]B, \neg B \in \Delta$. Consequently, for every $\Phi \in W^\circ$, if there exists $i \in (n)$ such that $\Delta \not\equiv_{ii} \Phi$ then $B \in \Phi$, $\neg B \not\in \Phi$ and $\Delta \neq \Phi$. Therefore, for every $\Delta, \Phi \in W^\circ$, $\Delta \neq \Phi$ iff there exists $i \in (n)$ such that $\Delta \not\equiv_{ii} \Phi$. Consequently, $\underline{W}^\circ$ is $\neq$-standard. Let it be proved that, for every $\Delta, \Phi \in W^\circ$ and for every $i \in (n)$, either $\Delta \equiv_{ii} \Phi$ or $\Delta \not\equiv_{ii} \Phi$. If neither $\Delta \equiv_{ii} \Phi$ nor $\Delta \not\equiv_{ii} \Phi$ then there exists a formula $[\equiv_{ii}]A \in \Delta$ such that $A \not\in \Phi$ and there exists a formula $[\not\equiv_{ii}]B \in \Delta$ such that $B \not\in \Phi$. Consequently, $[\equiv_{ii}](A \vee B) \wedge [\not\equiv_{ii}](A \vee B) \in \Delta$ and $\neg(A \vee B) \in \Phi$. Therefore, $[U](A \vee B) \in \Delta$ and $A \vee B \in \Phi$, a contradiction. Similarly, direct calculations would lead to the conclusion that, for every $\Delta, \Phi \in W^\circ$ and for every $i \in (n)$, either $\neg\Delta \equiv_{ii} \Phi$ or $\neg\Delta \not\equiv_{ii} \Phi$. Consequently, $\underline{W}^\circ$ is quasi-standard. Therefore:

Theorem 3 *BAL_n is complete for the class of all quasi-standard n-arrow models, that is to say: the formulas valid in every quasi-standard n-arrow model are theorems of BAL_n.*

4 Collinearity

Let $n = 3$. Collinearity is one of the basic ternary relations between the points of a geometrical structure.

4.1 Structures of collinearity

A *structure of collinearity* is a point 3-frame $\underline{S} = (P, C)$ such that:

- for every $X, Y \in P$, $C(X, Y, X)$,
- for every $X, Y, Z \in P$, $C(X, Y, Z)$ only if $C(Y, X, Z)$,
- for every $X, Y, Z, T \in P$, $C(X, Y, Z)$ and $C(X, Y, T)$ only if $X = Y$ or $C(X, Z, T)$.

The class of all structures of collinearity is denoted by Σ^C_3. The elements of P are called *points* and are denoted by capital letters. C is the relation of *collinearity* between the points of the relational structure.

Example 3 *The affine geometries axiomatized by Szczerba and Tarski [11] are structures of collinearity.*

4.2 Frames of collinearity

A *frame of collinearity* is a 3-arrow frame $\underline{W} = (W, \equiv_{ij})$ such that, for every $i, j, k, l \in (3)$:

- for every $x, y \in W$, there exists $u \in W$ such that $x \equiv_{i1} u$, $y \equiv_{j2} u$ and $x \equiv_{i3} u$,
- for every $x, y, z \in W$, there exists $u \in W$ such that $x \equiv_{i1} u$, $y \equiv_{j2} u$ and $z \equiv_{k3} u$ only if there exists $v \in W$ such that $y \equiv_{j1} v$, $x \equiv_{i2} v$ and $z \equiv_{k3} v$,
- for every $x, y, z, t \in W$, there exists $u \in W$ such that $x \equiv_{i1} u$, $y \equiv_{j2} u$ and $z \equiv_{k3} u$ and there exists $v \in W$ such that $x \equiv_{i1} v$, $y \equiv_{j2} v$ and $t \equiv_{l3} v$ only if either $x \equiv_{ij} y$ or there exists $w \in W$ such that $x \equiv_{i1} w$, $z \equiv_{k2} w$ and $t \equiv_{l3} w$.

The class of all normal frames of collinearity is denoted by Φ_3^C.

4.3 Structures and frames of collinearity

Direct calculations would lead to the conclusion that:

Lemma 13 *Let $\underline{S}$ be a structure of collinearity. Then $W(\underline{S})$ is a normal frame of collinearity.*

Lemma 14 *Let $\underline{W}$ be a normal frame of collinearity. Then $S(\underline{W})$ is a structure of collinearity.*

Therefore:

Theorem 4 *The categories Σ_3^C and Φ_3^C are equivalent.*

4.4 Modal logic of collinearity

This section introduces a modal logic with standard semantics in the class of all normal frames of collinearity. A *general frame of collinearity* consists of a non-empty set W together with, for every $i, j \in (3)$, a binary relation $\equiv_{ij}$ on W such that $(W, \equiv_{ij})$ is a frame of collinearity and, for every $i \in (3)$, a binary relation $\not\equiv_{ii}$ on W.

Lemma 15 *The quasi-standard frames of collinearity are modally definable in the class of all quasi-standard 3-arrow frames by the conjunction of the following schemata:*

- *for every $i, j \in (3)$, $< U > A \wedge B \rightarrow <\equiv_{i1}> (<\equiv_{2j}> A \wedge <\equiv_{3i}> B)$,*
- *for every $i, j, k \in (3)$, $<\equiv_{i1}> (<\equiv_{2j}> A \wedge <\equiv_{3k}> B) \rightarrow$ $<\equiv_{i2}> (<\equiv_{1j}> A \wedge <\equiv_{3k}> B)$,*
- *for every $i, j, k, l \in (3)$, $<\equiv_{i1}> (<\equiv_{2j}> (A \wedge$ $<\equiv_{j2}> (<\equiv_{1i}> B \wedge <\equiv_{3k}> C)) \wedge <\equiv_{3l}> D)$ $\rightarrow <\equiv_{ij}> A \vee <\neq> B \vee <\equiv_{i1}> (<\equiv_{2k}> C \wedge <\equiv_{3l}> D)$.*

Together with the axioms of BAL_3, all the instances of the previous schemata are axioms of BAL_3^C.

Theorem 5 *BAL_3^C is complete for the class of all quasi-standard models of collinearity.*

5 Sorted point n-frames and sorted n-arrow frames

Let $n \geq 2$. This section is devoted to the proof of the categorial equivalence between sorted point n-frames and sorted n-arrow frames.

5.1 Sorted point n-frames

A *sorted point n-frame* consists, for every $i \in (n)$, of a non-empty set S_i together with, for every $i, j \in (n)$ such that $i \neq j$, a binary relation R_{ij} between S_i and S_j such that:

- for every $i, j \in (n)$ such that $i \neq j$ and for every $X \in S_i$, there exists $Y \in S_j$ such that $X \ R_{ij} \ Y$,
- for every $i, j \in (n)$ such that $i \neq j$ and for every $X \in S_i, Y \in S_j$, if $X \ R_{ij} \ Y$ then there exists $Z_1 \in S_1, \ldots, Z_n \in S_n$ such that $X = Z_i$, $Y = Z_j$ and, for every $k, l \in (n)$ such that $k \neq l$, $X_k \ R_{kl} \ X_l$.

Lemma 16 *For every $i, j \in (n)$ such that $i \neq j$ and for every $X \in S_i, Y \in S_j$, if $X \ R_{ij} \ Y$ then $Y \ R_{ji} \ X$.*

The class of all sorted point n-frames is denoted by Σ^n and is considered as a category with morphisms the usual homomorphisms between relational structures.

5.2 Sorted n-arrow frames

The n-arrow frame $\underline{W} = (W, \equiv_{ij})$ is *sorted* if:

- for every $u_1, \ldots, u_n \in W$, if, for every $i, j \in (n)$ such that $i \neq j$, $u_i \equiv_{ii} \circ \equiv_{jj} u_j$ then there exists $u \in W$ such that, for every $i \in (n)$, $u_i \equiv_{ii} u$.

The class of all normal sorted n-arrow frames is denoted by Φ^n.

5.3 From sorted point n-frames to sorted n-arrow frames

Let $\underline{S} = (S_i, R_{ij})$ be a sorted point n-frame. Let $W = \{(X_1, \ldots, X_n)$: $X_1 \in S_1, \ldots, X_n \in S_n$ and, for every $i, j \in (n)$ such that $i \neq j$, $X_i \ R_{ij} \ X_j\}$. For every $i, j \in (n)$, let $\equiv_{ij}$ be the binary relation on W defined by $(X_1, \ldots, X_n) \equiv_{ij} (Y_1, \ldots, Y_n)$ iff $X_i = Y_j$.

Lemma 17 *$W(\underline{S}) = (W, \equiv_{ij})$ is a normal sorted n-arrow frame.*

Example 4 *It can be proved, for every $i, j \in (n)$ such that $i \neq j$, that if $S_i \cap S_j = \emptyset$ then $\equiv_{ij} = \emptyset$ and that if $S_i \subseteq S_j$ then $\equiv_{ij}$ is serial.*

5.4 From sorted n-arrow frames to sorted point n-frames

Let $\underline{W} = (W, \equiv_{ij})$ be a normal sorted n-arrow frame. For every $i \in (n)$, let $S_i = \{i(u)\colon\ u \in W\}$. For every $i, j \in (n)$ such that $i \neq j$, let R_{ij} be the binary relation between S_i and S_j defined by $i(u)\ R_{ij}\ j(v)$ iff $u \equiv_{ii} \circ \equiv_{jj} v$. Direct calculations would lead to the conclusion that:

Lemma 18 *$S(\underline{W}) = (S_i, R_{ij})$ is a sorted point n-frame.*

Example 5 *It can be proved, for every $i, j \in (n)$ such that $i \neq j$, that if $\equiv_{ij} = \emptyset$ then $S_i \cap S_j = \emptyset$ and that if $\equiv_{ij}$ is serial then $S_i \subseteq S_j$.*

5.5 Representation theorems

Let $\underline{W} = (W, \equiv_{ij})$ be a normal sorted n-arrow frame and $\underline{W}' = W(S(\underline{W}))$. For every $u \in W$, let $g(u) = (1(u), \ldots, n(u))$. Direct calculations would lead to the conclusion that g is an isomorphism from $\underline{W}$ into $\underline{W}'$. Therefore:

Lemma 19 *$\underline{W}$ and $\underline{W}'$ are isomorphic.*

Let $\underline{S} = (S_i, R_{ij})$ be a sorted point n-frame and $\underline{S}' = S(W(\underline{S}))$. For every $X \in S_i$, let $f(X) = \{(X_1, \ldots, X_n)\colon\ X_1 \in S_1, \ldots, X_n \in S_n,\ X = X_i$ and, for every $j, k \in (n)$ such that $j \neq k$, $X_j\ R_{jk}\ X_k\}$. Direct calculations would lead to the conclusion that f is an isomorphism from $\underline{S}$ into $\underline{S}'$. Consequently:

Lemma 20 *$\underline{S}$ and $\underline{S}'$ are isomorphic.*

Therefore:

Theorem 6 *The categories Σ^n and Φ^n are equivalent.*

6 Sorted arrow logic

This section introduces a modal logic with standard semantics in the class of all normal sorted n-arrow frames.

6.1 Semantics

Let $A_1, \ldots, A_n$ be formulas. Let $B_1 = true$ and, for every $k \geq 1$, $B_{k+1} = {<}\equiv_{(k+1)(k+1)}{>}{<}\equiv_{kk}{>}\ ([\neq]A_k \wedge B_k) \wedge \bigwedge_{i \in (k)} {<}\equiv_{(k+1)(k+1)}{>}{<}\equiv_{ii}{>}\ \neg A_i$. Let $\mathcal{M} = (W, \equiv_{ij}, \not\equiv_{ii}, m)$ be a quasi-standard n-arrow model. Direct calculations would lead to the conclusion that, for every $k \in (n)$ and for every $x \in W$, $x \models_{\mathcal{M}} B_k$ iff there exists $x_1, \ldots, x_k \in W$ such that $x_k = x$, $x_i \models_{\mathcal{M}} [\neq]A_i \wedge B_i$, for every $i \in (k-1)$ and $x_i \equiv_{ii} \circ \equiv_{jj} x_j$, for every $i, j \in (k)$ such that $i \neq j$. Consequently:

Lemma 21 *The quasi-standard sorted n-arrow frames are modally definable in the class of all quasi-standard n-arrow frames by the following schema:*

- $C_n = B_n \rightarrow <\equiv_{nn}> \bigwedge_{i \in (n-1)} <\equiv_{ii}> \neg A_i$.

Example 6 *B_2 is logically equivalent to the schema $<\equiv_{22}><\equiv_{11}> [\neq]A_1 \wedge <\equiv_{22}><\equiv_{11}> \neg A_1$. The schema B_3 is equivalent to $<\equiv_{33}><\equiv_{22}> ([\neq]A_2 \wedge B_2) \wedge <\equiv_{33}><\equiv_{11}> \neg A_1 \wedge <\equiv_{33}><\equiv_{22}> \neg A_2$.*

Together with the axioms of BAL_n, all the instances of the previous schema are axioms of BAL^n.

Theorem 7 *BAL^n is complete for the class of all quasi-standard sorted n-arrow models.*

7 Geometrical sorted point n-frames

Projective structures and orthogonal structures are examples of sorted point n-frames.

7.1 Projective geometry

Incidence is one of the basic binary relations between the points and the lines of a geometrical structure.

7.1.1 Projective structures

A *projective structure* is a relational structure $\underline{S} = (P, L, in)$ such that:

- $P \cap L = \emptyset$,
- for every $X, Y \in P$, there exists $x \in L$ such that $X\ in\ x$ and $Y\ in\ x$,
- for every $x, y \in L$, there exists $X \in P$ such that $X\ in\ x$ and $X\ in\ y$,
- for every $X, Y \in P$ and for every $x, y \in L$, $X\ in\ x$, $Y\ in\ x$, $X\ in\ y$ and $Y\ in\ y$ only if $X = Y$ or $x = y$.

The elements of L are called *lines* and are denoted by lower case letters. in is the relation of incidence between the points and the lines of the relational structure. The class of all projective structures is denoted by Σ^2_P.

Example 7 *The projective geometries axiomatized by Heyting [7] are projective structures.*

A projective structure (P, L, in) is considered as a sorted point 2-frame $(S_1, S_2, R_{12}, R_{21})$ where $S_1 = P$, $S_2 = L$, $R_{12} = in$ and $R_{21} = in^{-1}$.

7.1.2 Projective frames

A *projective frame* is a sorted 2-arrow frame $\underline{W} = (W, \equiv_{ij})$ such that:

- $\equiv_{12} = \emptyset$,
- for every $x, y \in W$, $x \equiv_{11} \circ \equiv_{22} \circ \equiv_{11} y$,
- for every $x, y \in W$, $x \equiv_{22} \circ \equiv_{11} \circ \equiv_{22} y$,

- for every $x, y, z, t \in W$, $x \equiv_{11} \circ \equiv_{22} z$, $y \equiv_{11} \circ \equiv_{22} z$, $x \equiv_{11} \circ \equiv_{22} t$ and $y \equiv_{11} \circ \equiv_{22} t$ only if $x \equiv_{11} y$ or $z \equiv_{22} t$.

The class of all normal projective frames is denoted by Φ^2_P.

7.1.3 Projective structures and projective frames

Direct calculations would lead to the conclusion that:

Lemma 22 *Let $\underline{S}$ be a projective structure. Then $W(\underline{S})$ is a normal projective frame.*

Lemma 23 *Let $\underline{W}$ be a normal projective frame. Then $S(\underline{W})$ is a projective structure.*

Therefore:

Theorem 8 *The categories Σ^2_P and Φ^2_P are equivalent.*

7.1.4 Projective modal logic

This section introduces a modal logic with standard semantics in the class of all normal projective frames. A *general projective frame* consists of a non-empty set W together with, for every $i, j \in (2)$, a binary relation $\equiv_{ij}$ on W such that $(W, \equiv_{ij})$ is a projective frame and, for every $i \in (2)$, a binary relation $\not\equiv_{ii}$ on W.

Lemma 24 *The quasi-standard projective frames are modally definable in the class of all quasi-standard sorted 2-arrow frames by the conjunction of the following schemata:*

- $[\equiv_{12}]false$,
- $[\equiv_{11}][\equiv_{22}][\equiv_{11}]A \rightarrow [U]A$,
- $[\equiv_{22}][\equiv_{11}][\equiv_{22}]A \rightarrow [U]A$,
- $<\equiv_{11}><\equiv_{22}> (A \wedge <\equiv_{22}><\equiv_{11}> ([\neq]B \wedge C)) \rightarrow$
  $[\equiv_{11}][\equiv_{22}](<\equiv_{22}> A \vee [\equiv_{22}][\equiv_{11}]B) \vee <\equiv_{11}> C$.

Together with the axioms of BAL^2, all the instances of the previous schemata are axioms of BAL^2_P.

Theorem 9 *BAL^2_P is complete for the class of all quasi-standard projective models.*

7.2 Orthogonal geometry

Orthogonality is one of the basic binary relations between the lines of a geometrical structure.

7.2.1 Orthogonal structures

An *orthogonal structure* is a relational structure $\underline{S} = (P, L, in, \perp)$ such that:

- $P \cap L = \emptyset$,
- for every $X, Y \in P$, there exists $x \in L$ such that $X \ in \ x$ and $Y \ in \ x$,
- for every $X, Y \in P$ and for every $x, y \in L$, $X \ in \ x$, $Y \ in \ x$, $X \ in \ y$ and $Y \ in \ y$ only if $X = Y$ or $x = y$,
- for every $X \in P$ and for every $x \in L$, there exists $y \in L$ such that $X \ in \ y$ and $x \perp y$,
- for every $X \in P$ and for every $x, y, z \in L$, $X \ in \ y$, $x \perp y$, $X \ in \ z$ and $x \perp z$ only if $y = z$,
- for every $x, y \in L$, $x \perp y$ only if $y \perp x$.

$\perp$ is the relation of orthogonality between the lines of the relational structure. The class of all orthogonal structures is denoted by Σ_O^3.

Example 8 *The orthogonal geometries axiomatized by Goldblatt [6] are orthogonal structures.*

An orthogonal structure $(P, L, in, \perp)$ is considered as a sorted point 3-frame $(S_1, S_2, S_3, R_{12}, R_{13}, R_{21}, R_{23}, R_{31}, R_{32})$ where $S_1 = P$, $S_2 = L$, $S_3 = L$, $R_{12} = in$, $R_{13} = in$, $R_{21} = in^{-1}$, $R_{23} = \perp$, $R_{31} = in^{-1}$ and $R_{32} = \perp^{-1}$.

7.2.2 Orthogonal frames

An *orthogonal frame* is a sorted 3-arrow frame $\underline{W} = (W, \equiv_{ij})$ such that, for every $i, j, k \in \{2, 3\}$:

- $\equiv_{12} = \emptyset$,
- $\equiv_{23}$ is serial,
- $\equiv_{32}$ is serial,
- for every $x, y \in W$, $x \equiv_{11} \circ \equiv_{23} y$ iff $x \equiv_{11} \circ \equiv_{33} y$,
- for every $x, y \in W$, $x \equiv_{11} \circ \equiv_{32} y$ iff $x \equiv_{11} \circ \equiv_{22} y$,
- for every $x, y \in W$, $x \equiv_{11} \circ \equiv_{ij} \circ \equiv_{11} y$,
- for every $x, y, z, t \in W$, $x \equiv_{11} \circ \equiv_{ii} z$, $y \equiv_{11} \circ \equiv_{ii} z$, $x \equiv_{11} \circ \equiv_{jj} t$ and $y \equiv_{11} \circ \equiv_{jj} t$ only if $x \equiv_{11} y$ or $z \equiv_{ij} t$,
- for every $x, y \in W$, $x \equiv_{11} \circ \equiv_{i2} \circ \equiv_{3j} y$,
- for every $x, y, z, t \in W$, $x \equiv_{11} \circ \equiv_{jj} z$, $y \equiv_{i2} \circ \equiv_{3j} z$, $x \equiv_{11} \circ \equiv_{kk} t$ and $y \equiv_{i2} \circ \equiv_{3k} t$ only if $z \equiv_{jk} t$,
- for every $x, y \in W$, $x \equiv_{i2} \circ \equiv_{3j} y$ only if $x \equiv_{i3} \circ \equiv_{2j} y$.

The class of all normal orthogonal frames is denoted by Φ_O^3.

7.2.3 Orthogonal structures and orthogonal frames

Direct calculations would lead to the conclusion that:

Lemma 25 *Let* $\underline{S}$ *be an orthogonal structure. Then* $W(\underline{S})$ *is a normal orthogonal frame.*

Lemma 26 *Let* $\underline{W}$ *be a normal orthogonal frame. Then* $S(\underline{W})$ *is an orthogonal structure.*

Therefore:

Theorem 10 *The categories* Σ^3_O *and* Φ^3_O *are equivalent.*

7.2.4 Orthogonal modal logic

A *general orthogonal frame* consists of a non-empty set W together with, for every $i, j \in \{1, 2, 3\}$, a binary relation $\equiv_{ij}$ on W such that $(W, \equiv_{ij})$ is an orthogonal frame and, for every $i \in (3)$, a binary relation $\not\equiv_{ii}$ on W.

Lemma 27 *The quasi-standard orthogonal frames are modally definable in the class of all quasi-standard sorted* 3*-arrow frames by the conjunction of the following schemata:*

- $[\equiv_{12}]false$,
- $<\equiv_{23}>\ true$,
- $<\equiv_{32}>\ true$,
- $[\equiv_{11}][\equiv_{33}]A \leftrightarrow [\equiv_{11}][\equiv_{23}]A$,
- $[\equiv_{11}][\equiv_{22}]A \leftrightarrow [\equiv_{11}][\equiv_{32}]A$,
- $[\equiv_{11}][\equiv_{ij}][\equiv_{11}]A \rightarrow [U]A$,
- $<\equiv_{11}><\equiv_{ii}>\ (A\wedge <\equiv_{ii}><\equiv_{11}>\ ([\neq]B \wedge C)) \rightarrow [\equiv_{11}][\equiv_{jj}](<\equiv_{ji}>\ A \vee [\equiv_{jj}][\equiv_{11}]B)\vee <\equiv_{11}>\ C$,
- $[\equiv_{11}][\equiv_{i2}][\equiv_{3j}]A \rightarrow [U]A$,
- $<\equiv_{11}><\equiv_{jj}>\ (A\wedge <\equiv_{j3}><\equiv_{2i}>\ [\neq]B) \rightarrow [\equiv_{11}][\equiv_{kk}](<\equiv_{kj}>\ A \vee [\equiv_{k3}][\equiv_{2i}]B)$,
- $[\equiv_{i3}][\equiv_{2j}]A \rightarrow [\equiv_{i2}][\equiv_{3j}]A$.

Together with the axioms of BAL^3, all the instances of the previous schemata are axioms of BAL^3_O:

Theorem 11 BAL^3_O *is complete for the class of all quasi-standard orthogonal models.*

8 Conclusion

The methodology presented in this paper for a modal treatment of structures of collinearity, projective structures and orthogonal structures can be applied to other geometrical structures as well:

- Structures of betweeness consist of a set of points together with a ternary relation of betweeness between points [11].

- Projective structures of space consist of a set of points, a set of lines and a set of planes together with three binary relations of incidence between points, lines and planes.

However, many questions remain unsolved:

- Proof of the completeness or the incompleteness of BAL_3^C, BAL_P^2 and BAL_O^3 without the irreflexivity rule.
- Proof of the decidability or the undecidability of BAL_3^C, BAL_P^2 and BAL_O^3.

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