

Quantum in Pictures Lecture Series

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Wed 28 June 2023 – Afternoon Lecture



INDIANA UNIVERSITY BLOOMINGTON

$$\mathbb{C}[\mathbb{Z}_d] \quad (|j\rangle)_{j=0, \dots, d-1}$$

Comp basis

$$(|\chi\rangle)_{\kappa=0, \dots, d-1}$$

Fourier basis

$$|\chi_\kappa\rangle = \frac{1}{\sqrt{d}} \sum_{j=0}^{d-1} \chi_\kappa(j) |j\rangle$$

$$\chi_\kappa(j) = e^{i 2\pi \frac{\kappa \cdot j}{d}} \in S^1$$

Mult. character

$$\chi_\kappa : \mathbb{Z}_d \rightarrow S^1$$

$$\chi_\kappa(i+j) = \chi_\kappa(i) \cdot \chi_\kappa(j)$$

$$\chi_\kappa(0) = 1$$

$$\chi_\kappa(-j) = \chi_\kappa(j)^*$$

$$2\pi \frac{(k+ad)(j+bd)}{d} =$$

$$= 2\pi \frac{\kappa j}{d} + 2\pi \frac{\boxed{\kappa} \cancel{a} \cancel{b} d}{d} \stackrel{\text{mod } 2\pi}{=} 2\pi \frac{\kappa j}{d}$$

$$\mathbb{Z}_n \longleftrightarrow \mathbb{Z}_n^*$$

$$j \longmapsto x_j$$

$$j \longmapsto x_{pj}$$

$$p \in \mathbb{Z}_n^*$$

$$\begin{array}{l}
 \mathbb{R} \longleftrightarrow \mathbb{R}^1 \\
 \lambda \longmapsto (y \mapsto e^{i\lambda y}) \\
 \lambda \longmapsto (y \mapsto e^{i\frac{\lambda}{h}y}) \quad h \neq 0
 \end{array}$$

$$\underbrace{\left(\begin{array}{c} \vdots \\ \vdots \\ \vdots \end{array} \right)}_n \underbrace{\left(\begin{array}{c} \vdots \\ \vdots \\ \vdots \end{array} \right)}_m \underset{\alpha}{=} \sum_{j=0}^{d-1} e^{i\alpha_j} \underbrace{|j \dots j\rangle}_m \underbrace{\langle j \dots j|}_n$$

$$\alpha_0 = 0$$

$$\left(e^{i\alpha_1} e^{i\alpha_2} \dots e^{i\alpha_{d-1}} \right)$$

$$\underbrace{\left(\begin{array}{c} \vdots \\ \vdots \\ \vdots \end{array} \right)}_n \underbrace{\left(\begin{array}{c} \vdots \\ \vdots \\ \vdots \end{array} \right)}_m \underset{\alpha}{=} \sum_{j=0}^{d-1} e^{i\alpha_j} \underbrace{|\chi_j \dots \chi_j\rangle}_m \underbrace{\langle \chi_j \dots \chi_j|}_n$$

$$\underbrace{\left(\begin{array}{c} | \\ | \\ | \end{array} \right)}_n \underset{\alpha}{=} \sum_{j=0}^{d-1} |\chi_j\rangle \langle j|$$

$$\underbrace{\left(\begin{array}{c} | \\ | \\ | \end{array} \right)}_n \underset{\alpha}{=} \sum_{j=0}^{d-1} |j\rangle \langle \chi_j|$$

$$\begin{pmatrix} a & b & c \\ \lambda & e & f \\ \gamma & h & k \end{pmatrix}^T$$

$$\begin{pmatrix} a & d & g \\ b & e & h \\ c & f & k \end{pmatrix}$$

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$|1\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$|2\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

for $d > 2$

$$\left(\right) := \left\{ \right. = \sum_{j=0}^{d-1} |j\rangle \neq \sum_{j=0}^{d-1} |x_j x_j\rangle = \left(\oplus \right)$$

$$\left. \right) = \left(\right) = \sum_{j=0}^{d-1} \text{انداك من}$$

$$-\boxed{H} - \boxed{H} = \sum_{j'=0}^{d-1} \sum_{j=0}^{d-1} \langle j' | x_j \rangle \langle z | \underbrace{\langle x_j | z \rangle}_{(\langle x_j | z \rangle)^t} = x_j(z)^t = x_{j(-j)}$$

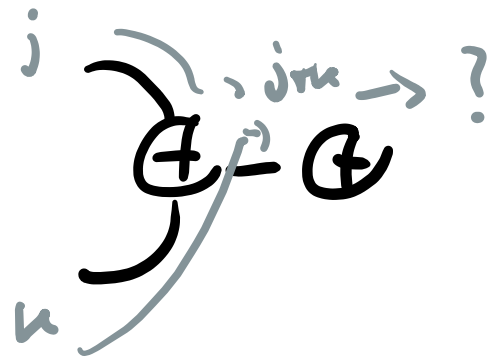
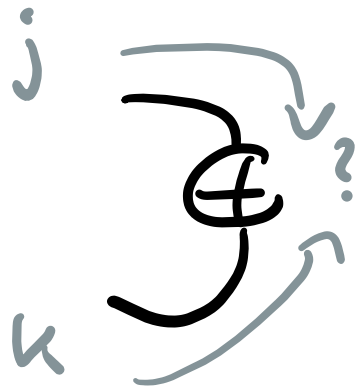
$$\sum_{j'=0}^{d-1} x_j(z') = \begin{cases} 0 & \text{if } j \neq 0 \\ 1 & \text{if } j = 0 \end{cases} = : -\boxed{H} = \text{antipode} = \text{group inverse}$$

$$-|j\rangle = -|j\rangle - |j\rangle$$

$$-|j\rangle - |j\rangle = |j\rangle \mapsto |-j\rangle \mapsto | -(-j) \rangle = |j\rangle = \text{---}$$

i.e. $-|j\rangle = -|j\rangle - |j\rangle = -|j\rangle - |j\rangle = -|j\rangle - |j\rangle - |j\rangle$

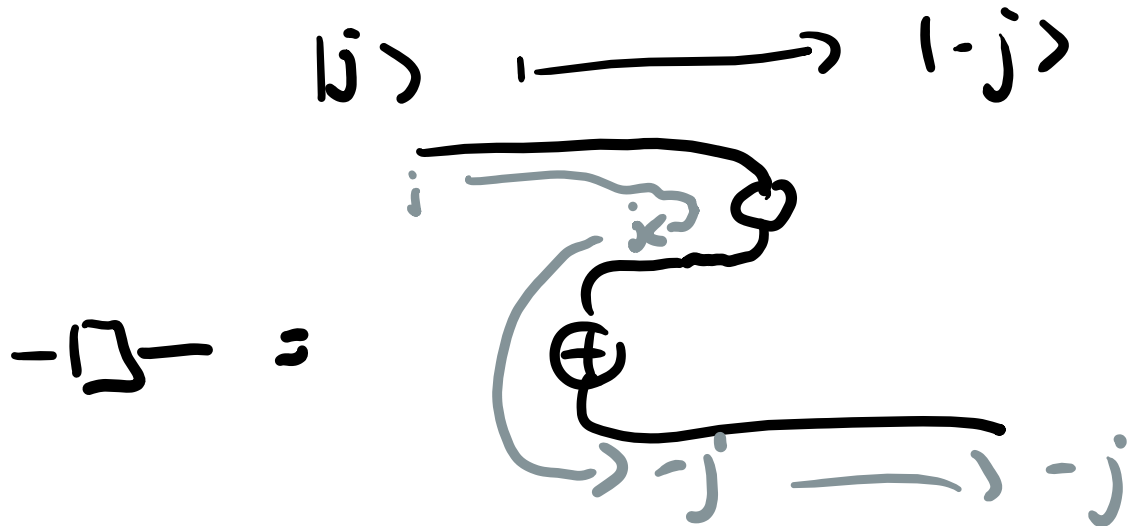
$d=2$ $-|j\rangle = |j\rangle \mapsto |-j\rangle \stackrel{\substack{\uparrow \\ = 1 \text{ mod } 2}}{=} |j\rangle \Rightarrow -|j\rangle - |j\rangle = \text{---}$



$$-\oplus = (\oplus -)^+ = (10)^+ = \langle 0 |$$

$$\oplus = |j^k\rangle \mapsto \begin{cases} 1 & j = -k \\ 0 & \text{otherwise} \end{cases} \in \mathbb{C}$$

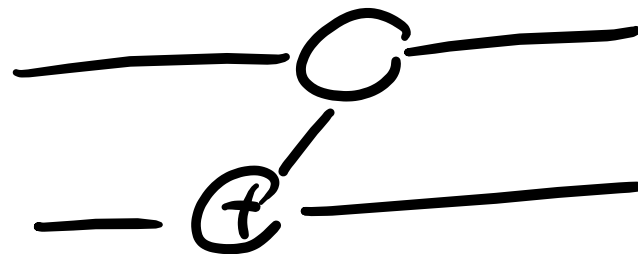
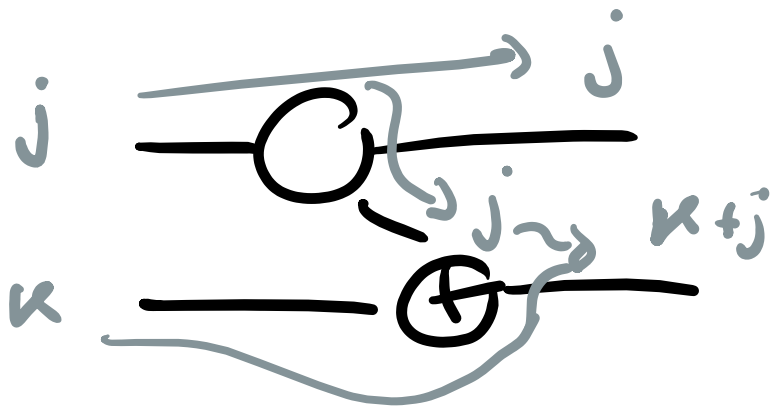
$$\mathbb{C} \oplus = \sum_j |j, -j\rangle$$



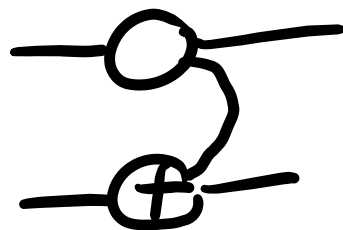
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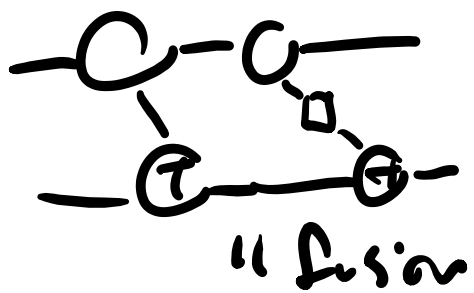
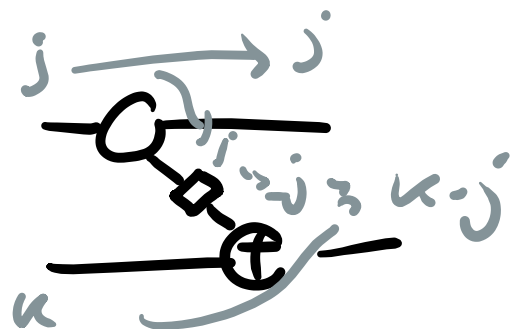
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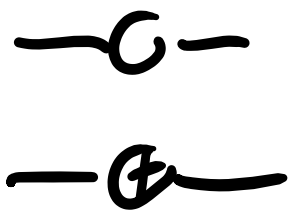
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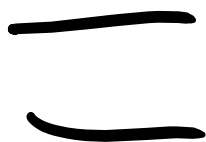
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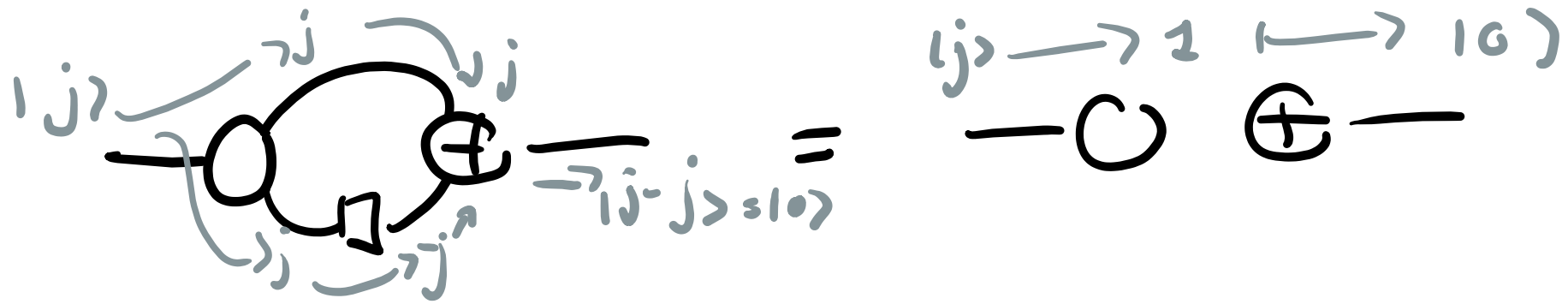


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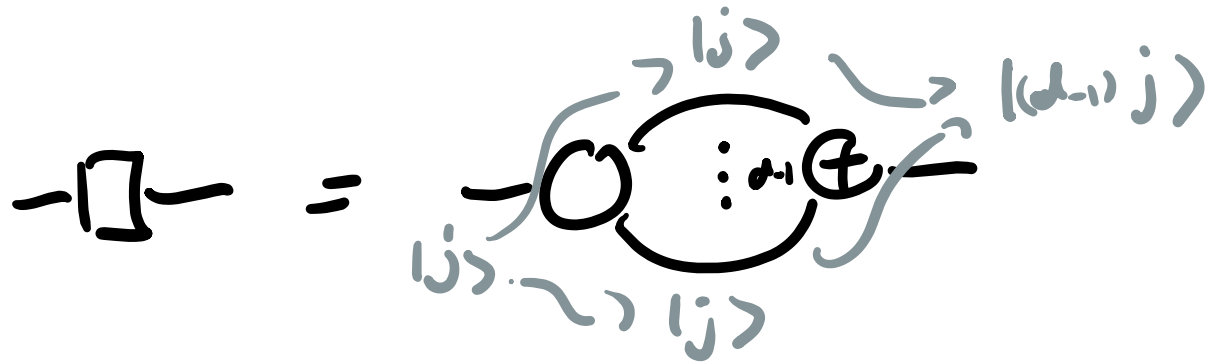


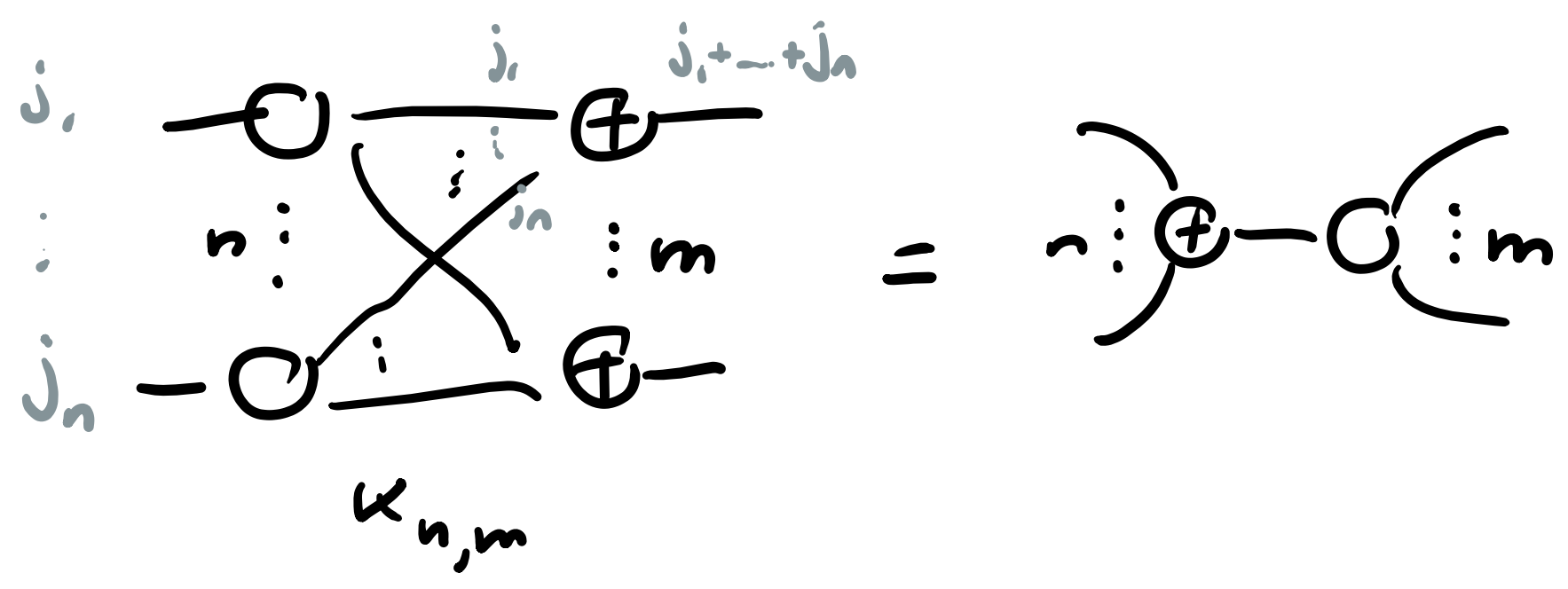
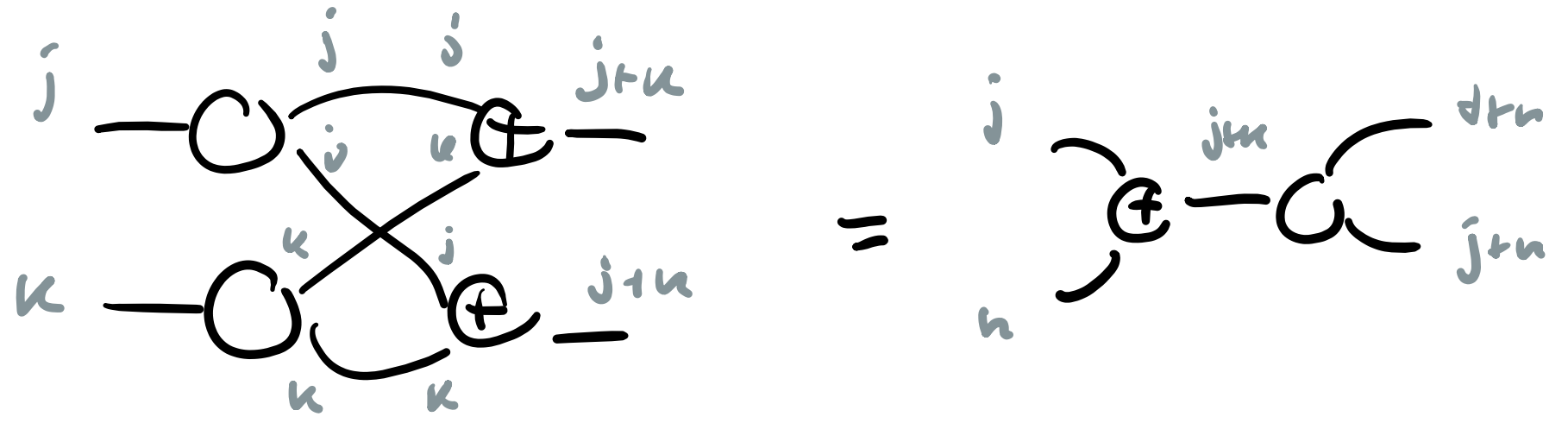
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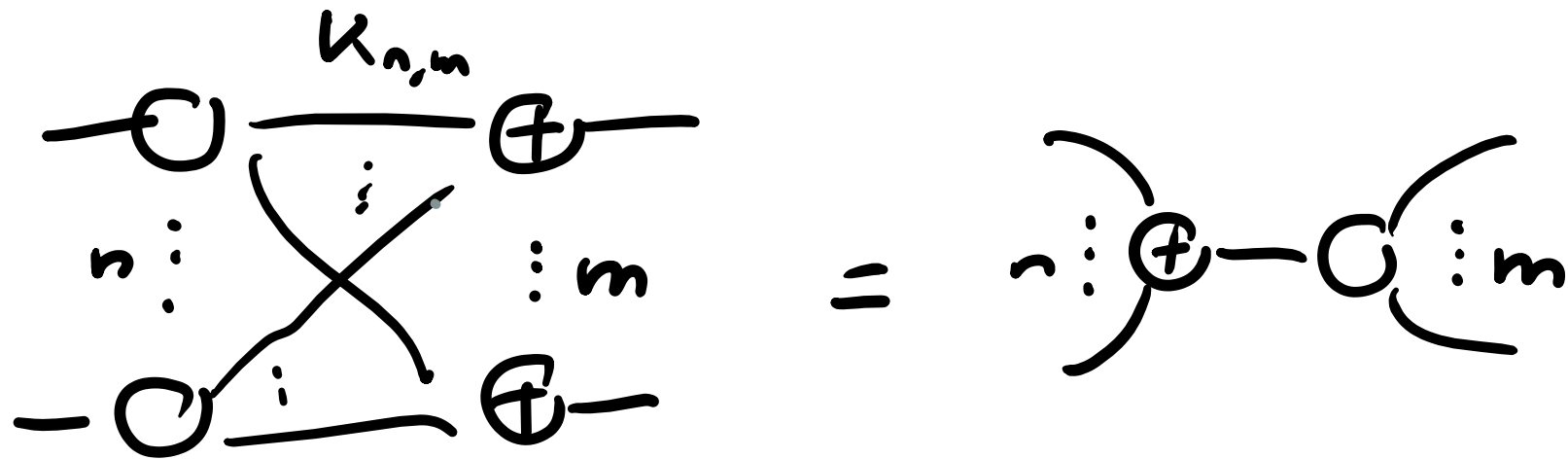




$$\text{mod } d \quad -j = (d-1)j$$



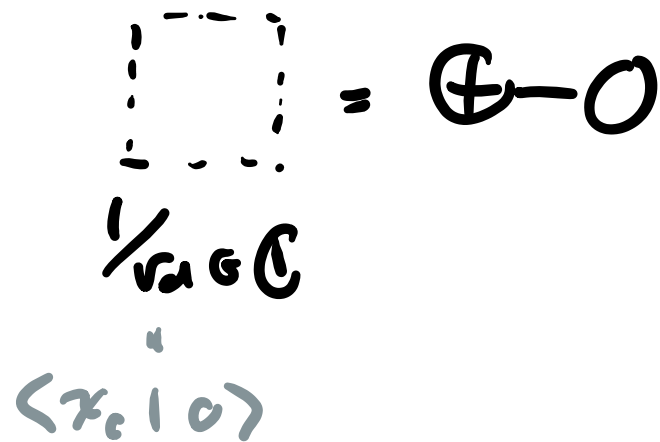
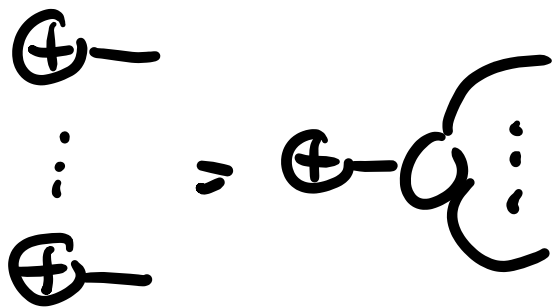
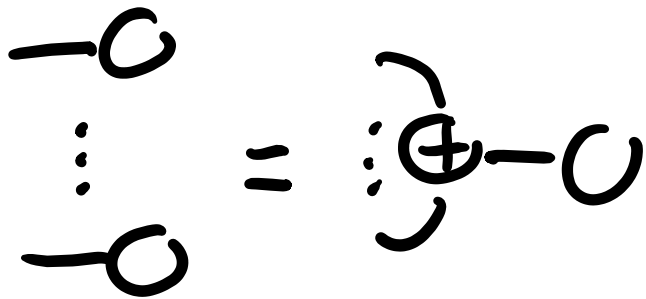




$$m=0, n>0$$

$$n=0, m>0$$

$$n=m=0$$



$$\bigoplus_{\alpha} - = |\alpha_0\rangle + e^{i\alpha_1} |\alpha_1\rangle + e^{i\alpha_2} |\alpha_2\rangle + \dots + e^{i\alpha_{d-1}} |\alpha_{d-1}\rangle$$

$$\alpha_j := 2\pi \frac{-jk}{d}$$

$$\bigoplus_{\alpha} - = |\alpha_0\rangle + e^{-i2\pi \frac{k}{d}} |\alpha_1\rangle + e^{-i2\pi \frac{2k}{d}} |\alpha_2\rangle \dots = |k\rangle$$

$$\bigoplus_{[j]} - = |j\rangle \quad [j]_k := 2\pi \frac{-jk}{d}$$

$$d=2$$

$$\bigoplus - = |0\rangle$$

$$\bigoplus_{\pi} - = |1\rangle$$

$$\oplus \text{---} = |j\rangle$$

$[j]$

$$[j]_k = 2\pi \frac{-jk}{\alpha}$$

$$\bigcirc \text{---} = |k\rangle$$

$[k]$

$$[k]_j = 2\pi \frac{kj}{\alpha}$$

$$\oplus \text{---} \bigcirc = \begin{matrix} \oplus \text{---} \\ \oplus \text{---} \end{matrix}$$

$[j]$ $[j]$

$$\oplus \text{---} \bigcirc = 1$$

$[j]$

$$\bigcirc \text{---} \oplus = \begin{matrix} \text{---} \bigcirc \\ \text{---} \bigcirc \end{matrix}$$

$[k]$ $[k]$

$$\oplus \text{---} \bigcirc = 1$$

$[k]$

$$\begin{array}{c} a \\ \downarrow \\ \oplus - \bigcirc [u] \\ \downarrow \\ \oplus - \bigcirc [u] = \begin{array}{c} a \rightarrow \bigcirc [u] \\ b \rightarrow -\bigcirc [u] \end{array}
 \end{array}$$

$$\chi_u(ab) = \chi_u(a) \cdot \chi_u(b)$$

$$\oplus - \bigcirc [u] = 1$$

$$\chi_u(\emptyset) = 1$$

$$\begin{array}{c} \bigcirc [u] \\ \downarrow \\ \bigcirc [u] = \bigcirc [u] = \left(-\bigcirc [u] \right)^{\dagger}
 \end{array}$$

$$\chi_{-u}(j) = \chi_u(j)^{\dagger}$$

copy/delete/transpose
for $\bigcirc [u]$



mult character
 $\pi_a \cong S^1$

—○—
[α]

↓
 \mathbb{Z}_d

Z_α

—⊕—
[j]

↓
 \mathbb{Z}_d

X_j

—○—⊕—
[α] [j]

$\gamma_{\alpha,j}$

Weyl CCZ

$$\begin{array}{cccc}
 \text{---} & \text{---} & \text{---} & \text{---} \\
 \text{C} & \oplus & \text{C} & \oplus \\
 [a] & [b] & [c] & [d]
 \end{array}$$

$$= \begin{array}{cc}
 \oplus & \text{---} \\
 [a] & \text{C} \\
 [b] & [c]
 \end{array}
 \begin{array}{cc}
 \text{---} & \text{---} \\
 \text{C} & \oplus \\
 [a+c] & [b+d]
 \end{array}$$

$\leftarrow e^{i2\pi \frac{bc}{d}}$

$$\begin{array}{c}
 \oplus \\
 \text{---} \\
 \text{C} \\
 [c]
 \end{array}
 = \begin{array}{c}
 \text{---} \\
 \text{C} \\
 [c]
 \end{array}
 \begin{array}{c}
 \oplus \\
 \text{---} \\
 \text{C} \\
 [c]
 \end{array}$$

$$\begin{array}{cc}
 \oplus & \text{---} \\
 [b] & \text{C} \\
 & [c]
 \end{array}
 \begin{array}{c}
 \oplus \\
 \text{---} \\
 \text{C} \\
 [c]
 \end{array}$$

$$= \begin{array}{cc}
 \text{---} & \oplus \\
 \text{C} & \text{---} \\
 [b] & [c]
 \end{array}
 \begin{array}{cc}
 \oplus & \text{---} \\
 \text{---} & \text{C} \\
 [b] & [c]
 \end{array}$$

+ fusion

$$\begin{array}{cc}
 \text{---} & \text{---} \\
 \text{C} & \oplus \\
 [c] & [b]
 \end{array}
 \begin{array}{cc}
 \oplus & \text{---} \\
 \text{---} & \text{C} \\
 [b] & [c]
 \end{array}$$

$$\begin{array}{c}
 \oplus \\
 \text{---} \\
 \text{C} \\
 [c]
 \end{array}
 \text{---}
 \text{C}
 = \begin{array}{cc}
 \text{---} & \oplus \\
 \text{C} & \text{---} \\
 [c] & [c]
 \end{array}$$

$$\begin{array}{c}
 \oplus \\
 \text{---} \\
 \text{C} \\
 [b]
 \end{array}
 \text{---}
 \text{C}
 = \begin{array}{cc}
 [b] & \oplus \\
 [b] & \oplus
 \end{array}$$

$$Y_{a,b} = \alpha \frac{-C - \ominus}{[a] [b]}$$

$$(Y_{a,b})^d = s^{\alpha^d} \frac{-\ominus - \oplus}{[a] [b]} \dots \frac{-\oplus - \ominus}{[a] [b]}$$

$\xrightarrow{\alpha-1}$ $\xrightarrow{2}$ $\xrightarrow{2}$

$$e^{i 2\pi \frac{ab}{a}}$$

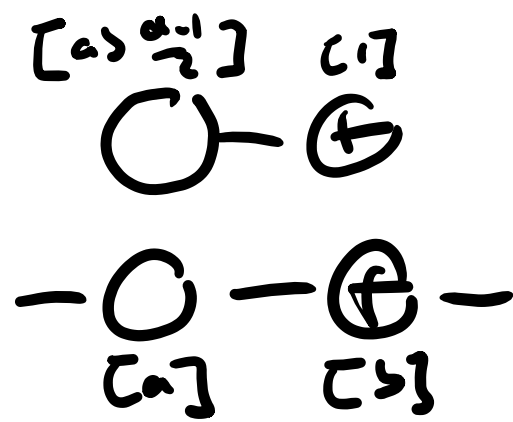
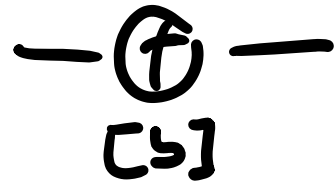
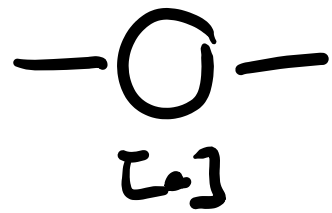
$$\alpha^d \left(e^{i 2\pi \frac{ab}{a}} \right)^{\frac{(d-1)\alpha}{2}}$$

$$\underbrace{\hspace{10em}}_1$$

$$\alpha = e^{-i 2\pi \frac{ab (d-1)}{2}}$$

$$\begin{aligned} a &= 1 \\ b &= 1 \Rightarrow \alpha = \pm i \\ d &= 2 \end{aligned}$$

$$\frac{1}{2} \in \mathbb{Z} \alpha^x$$



Z_a

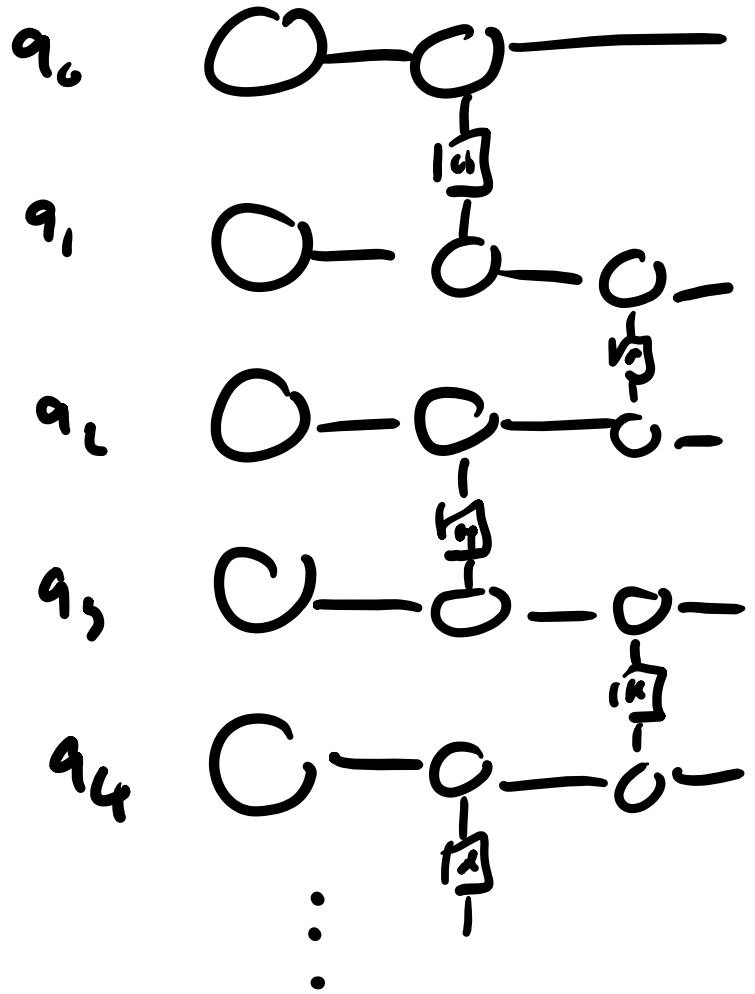
X_b

$Y_{a,b}$

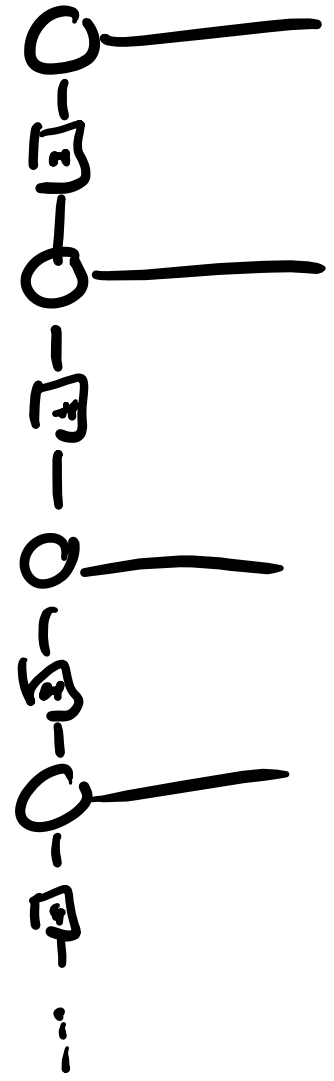
Pauli group in $\mathbb{C}[Z_a]$

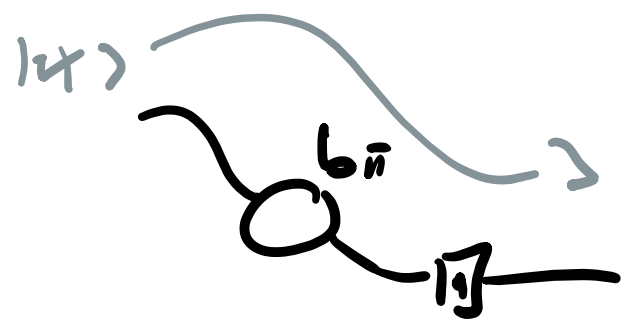
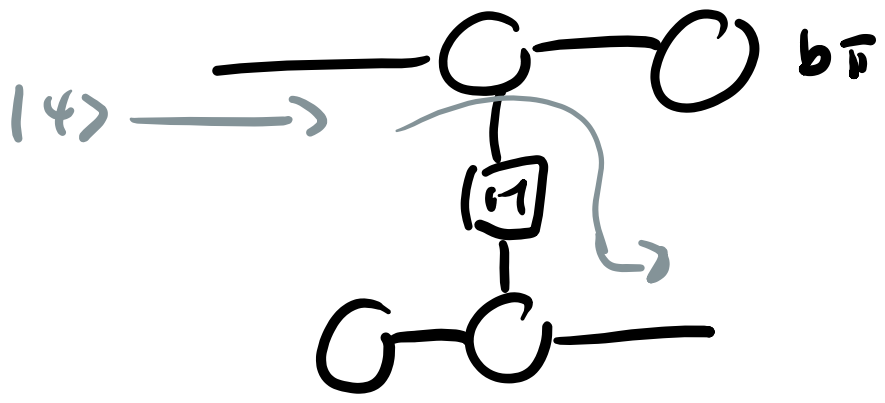
Z_x for qudit stabilizers:

1D Cluster State



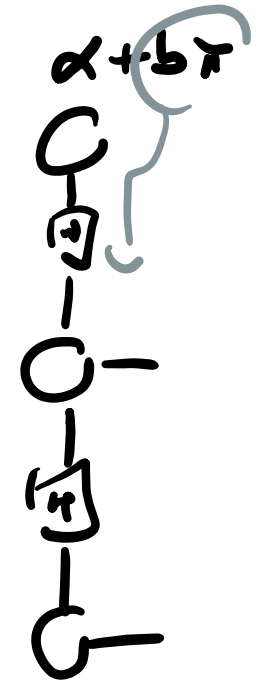
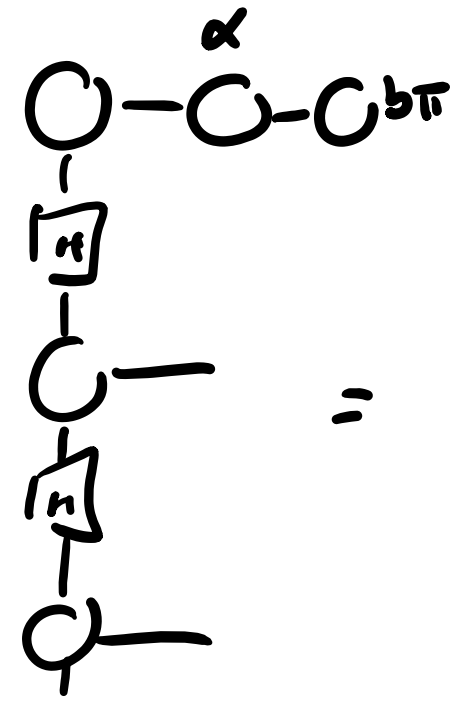
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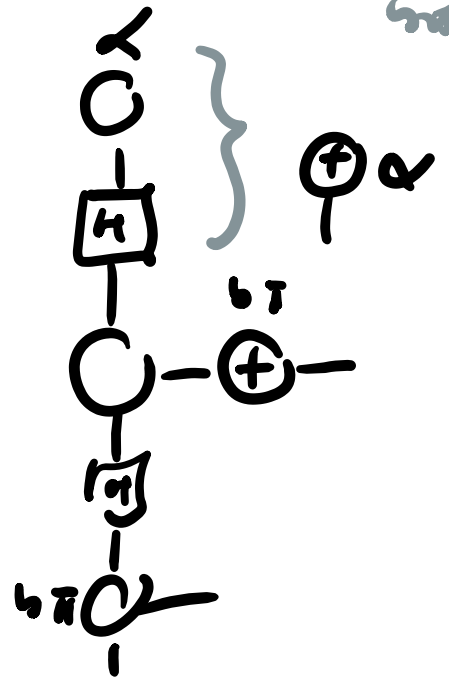
Corrector

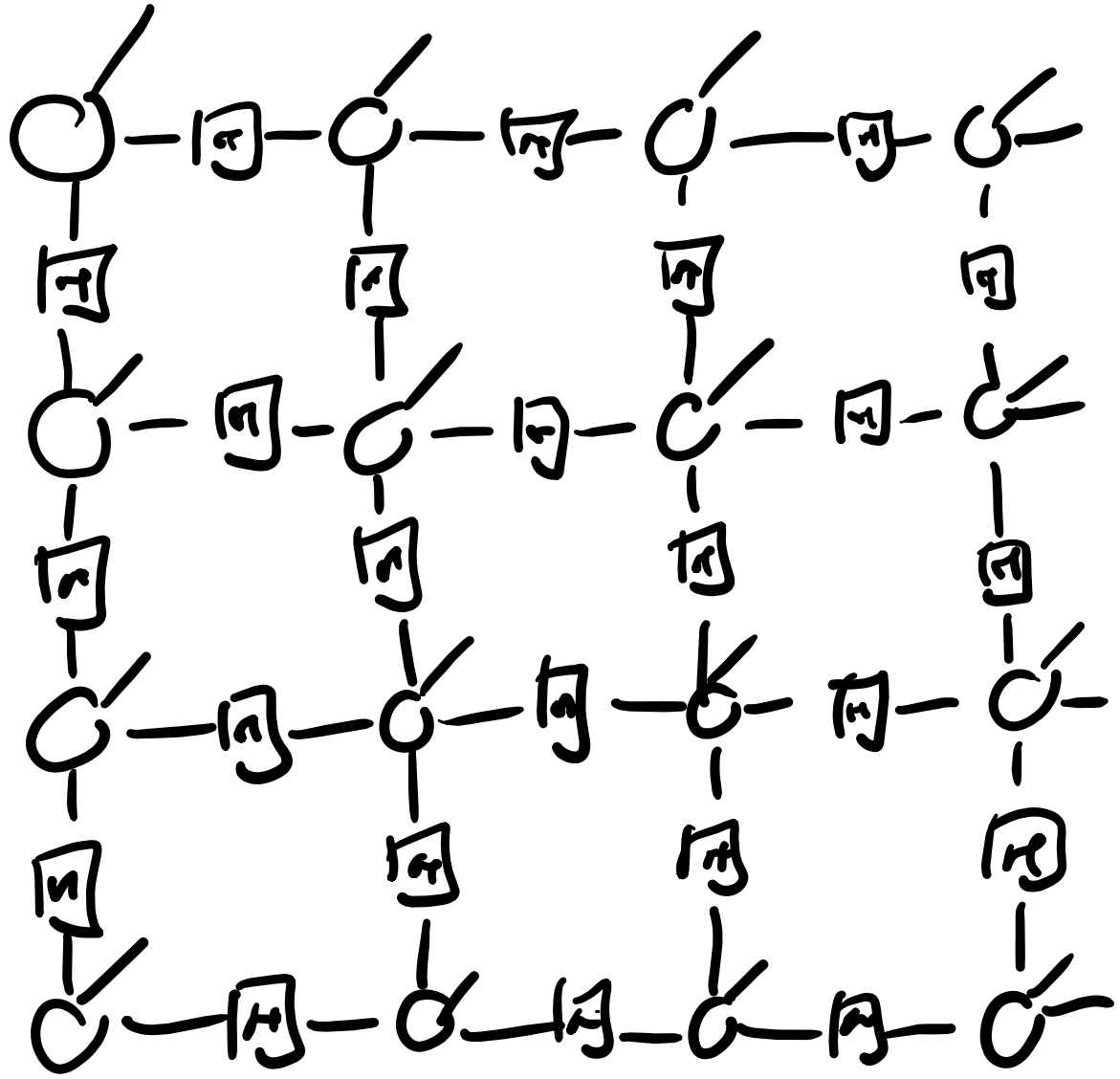


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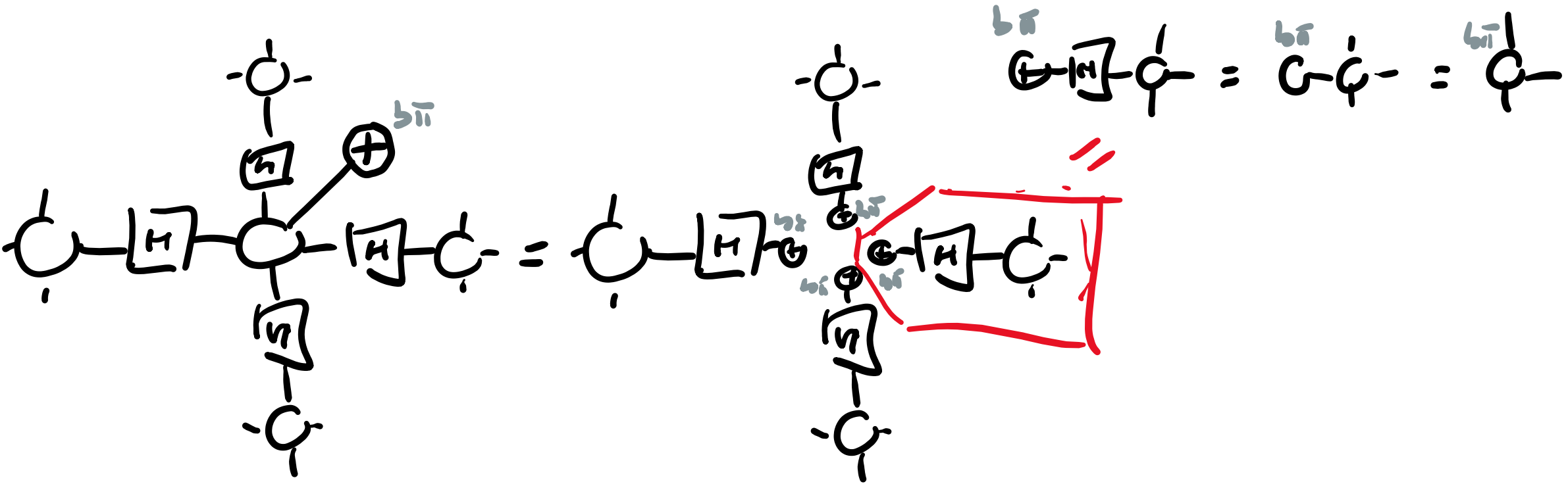
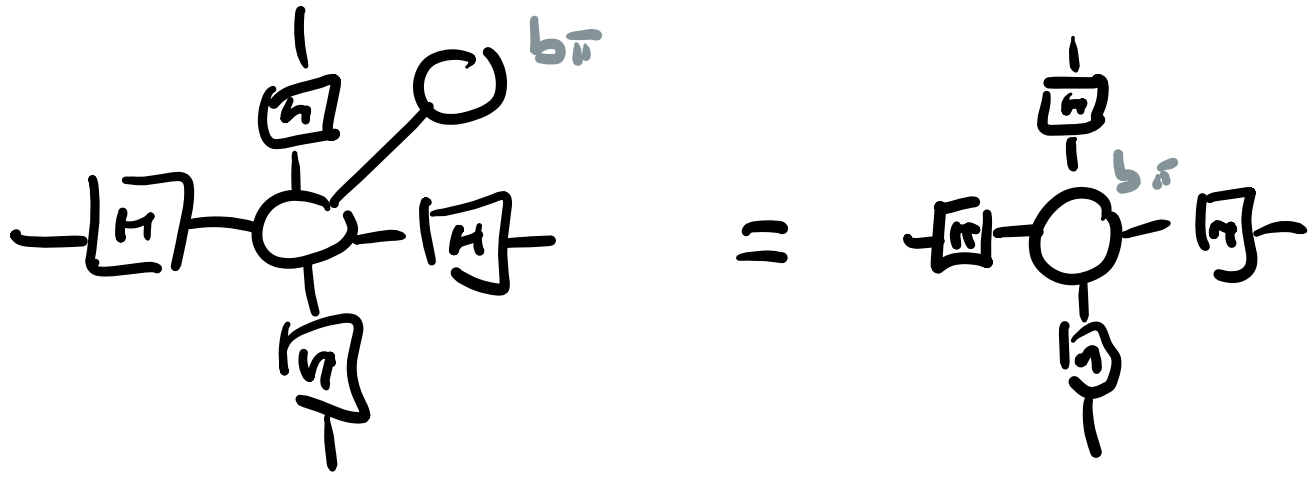


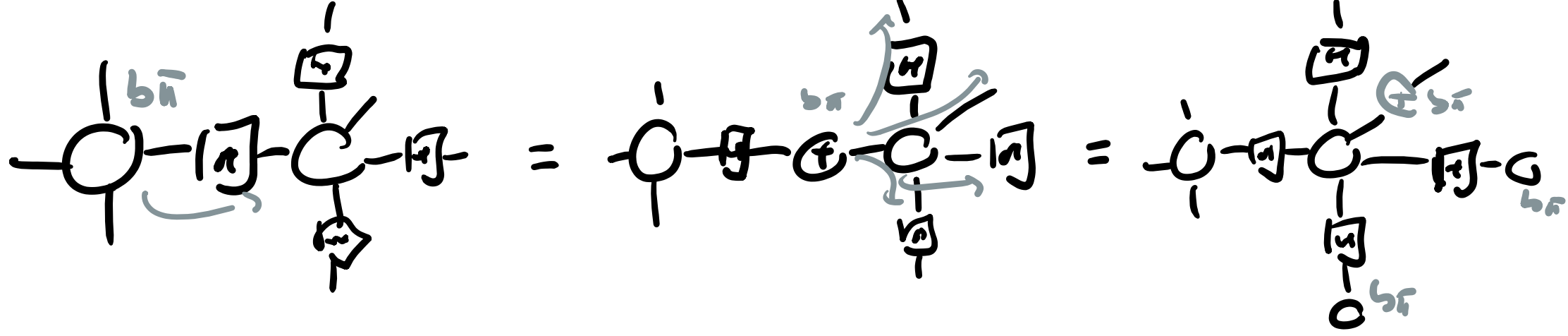
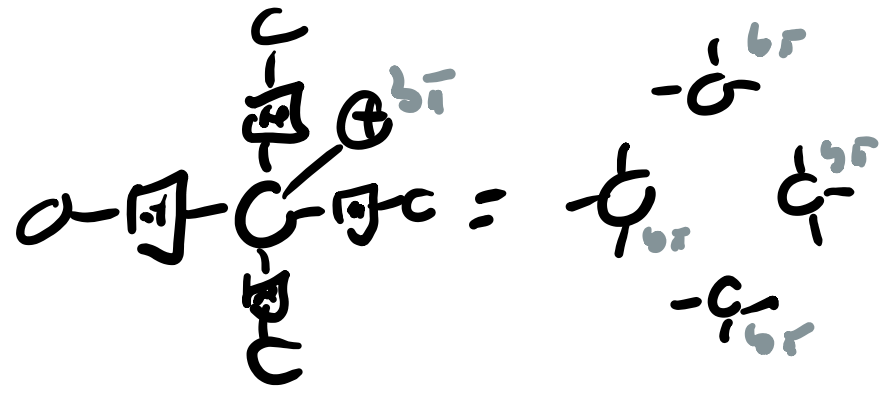
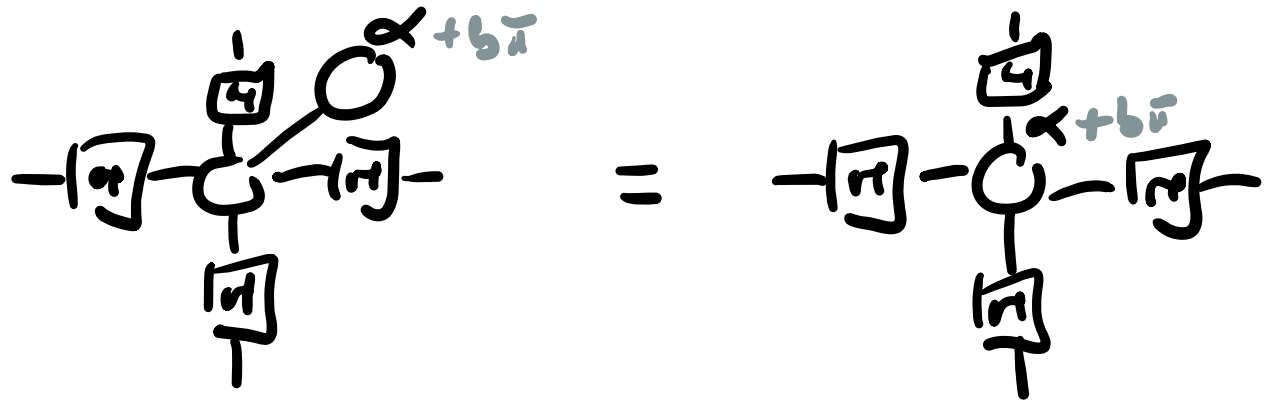
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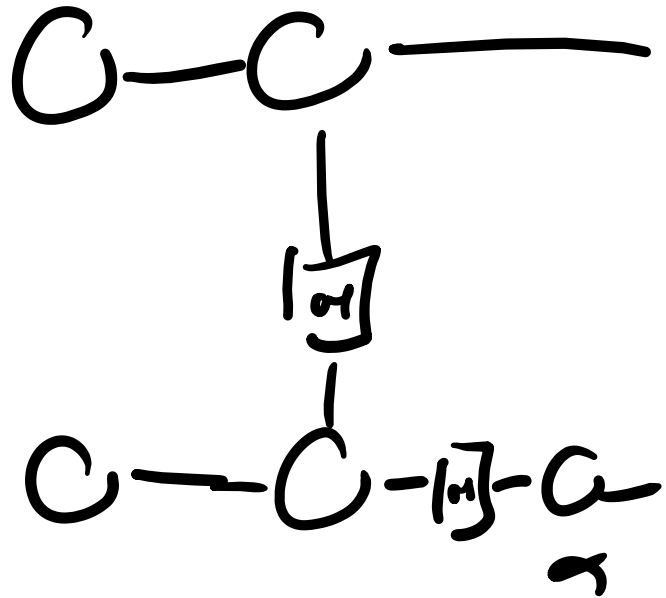




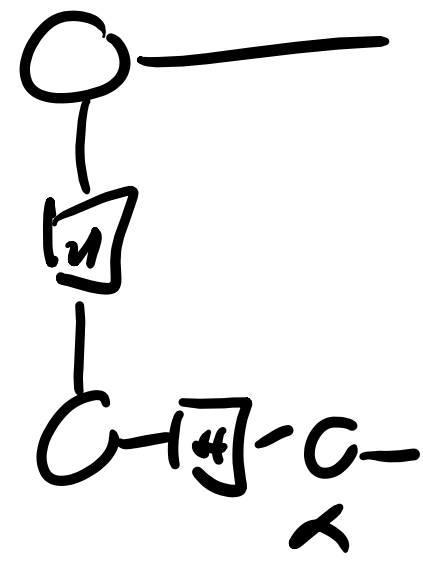
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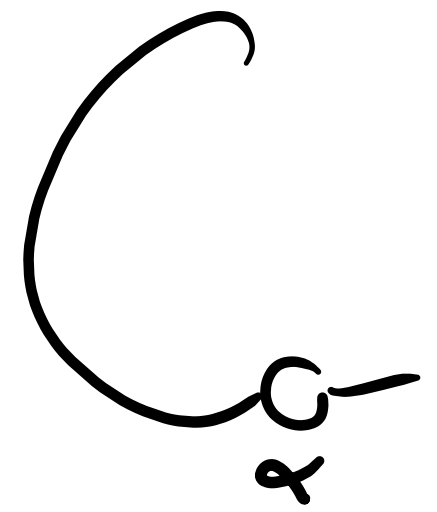


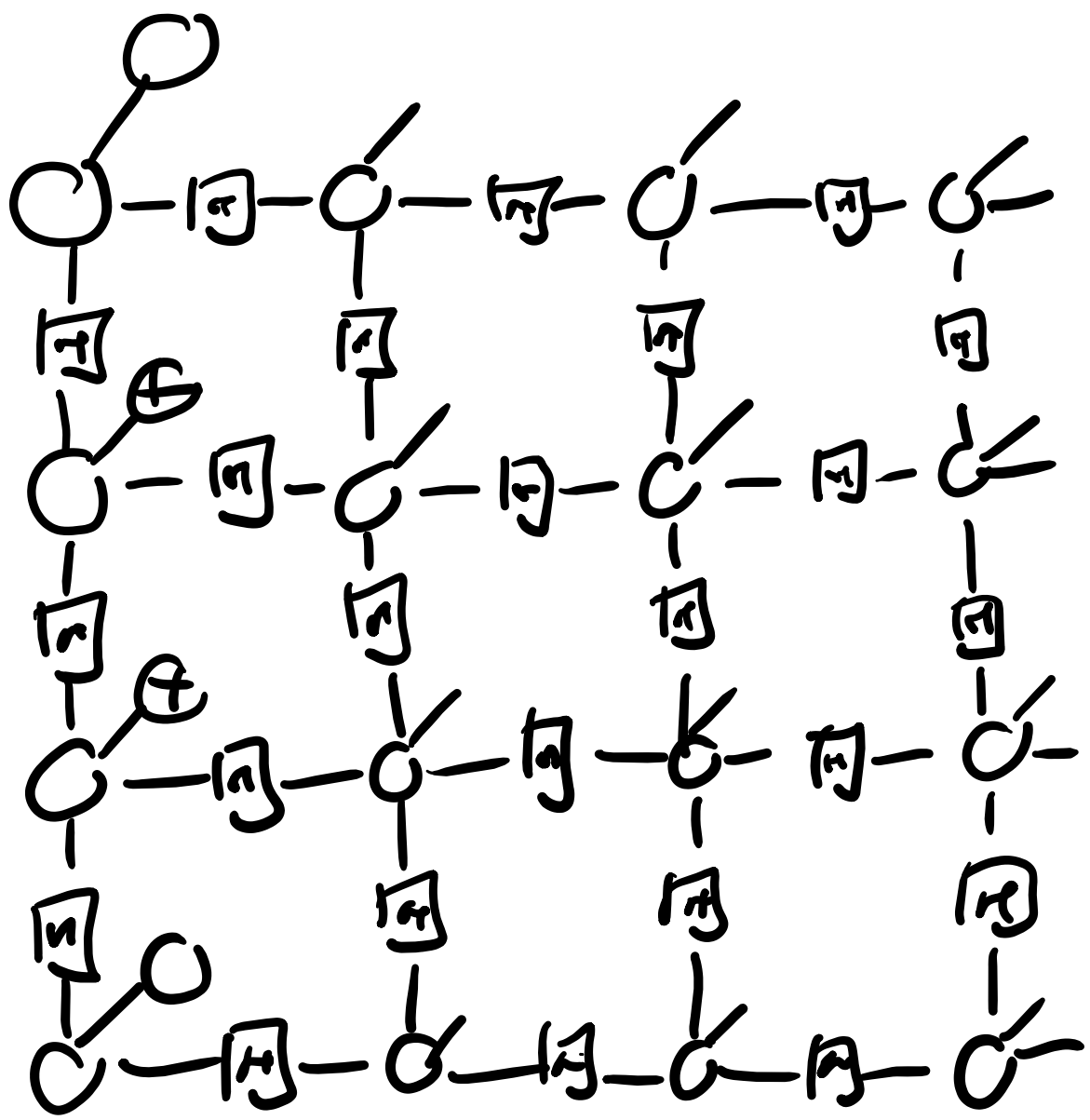


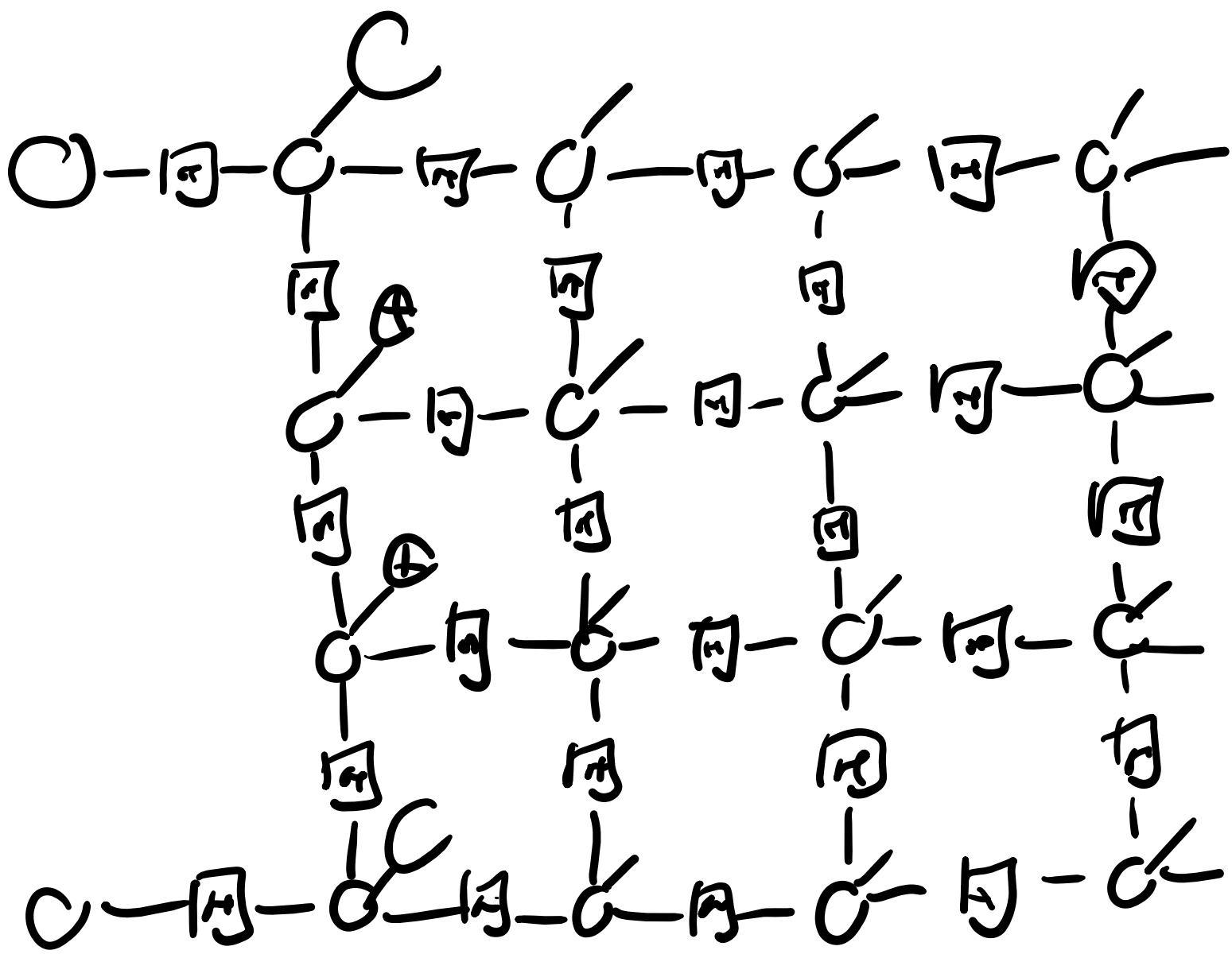
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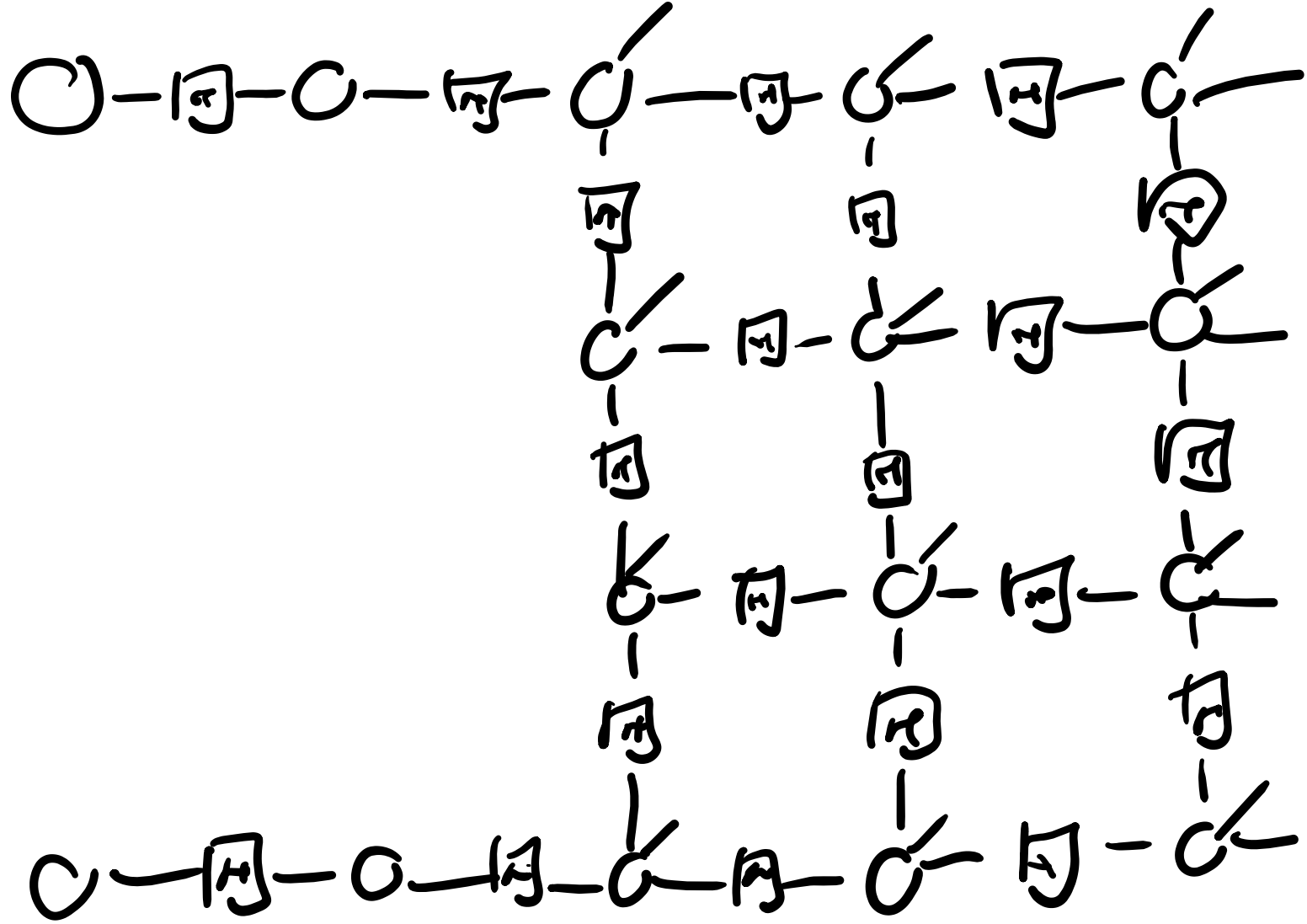


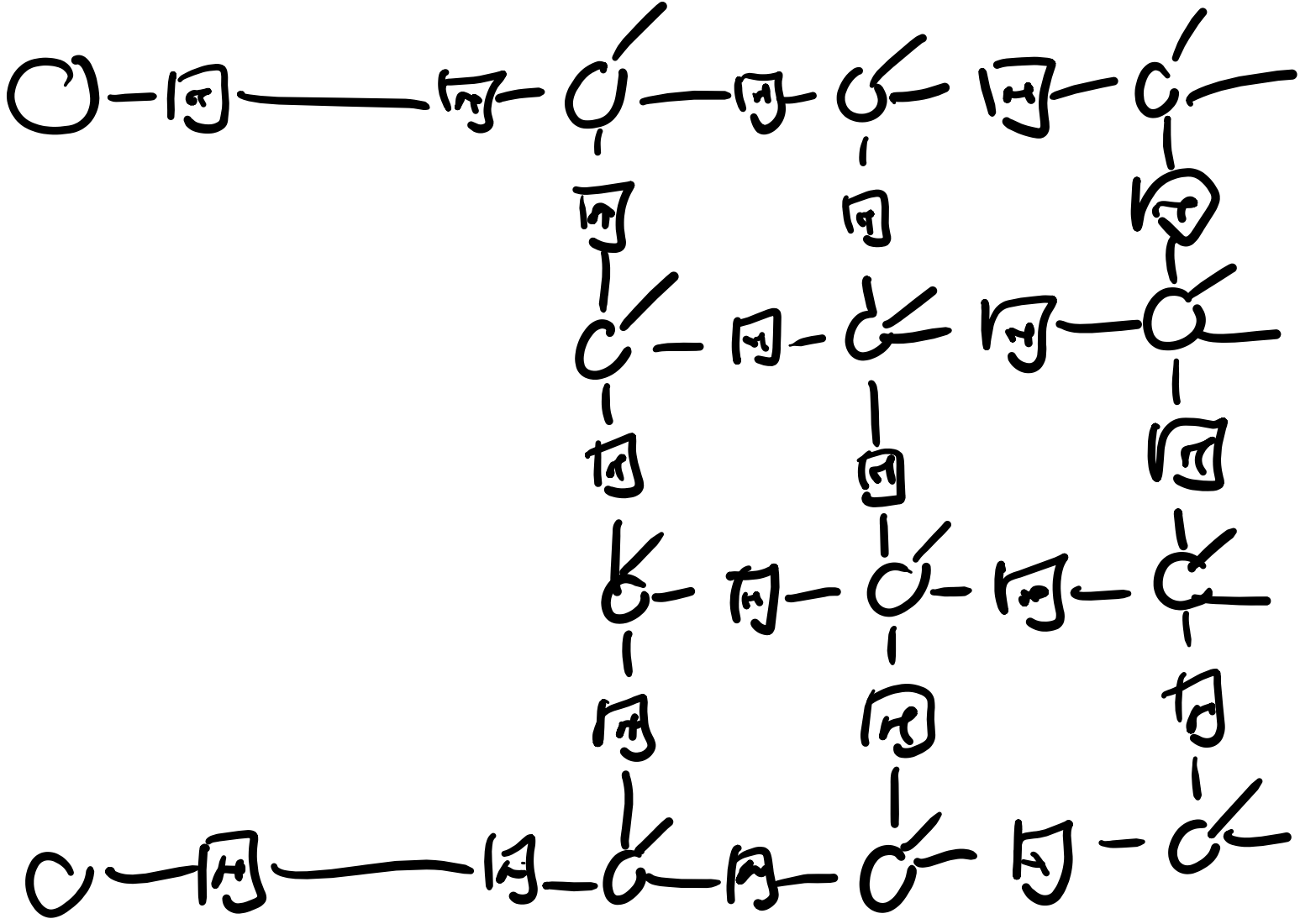
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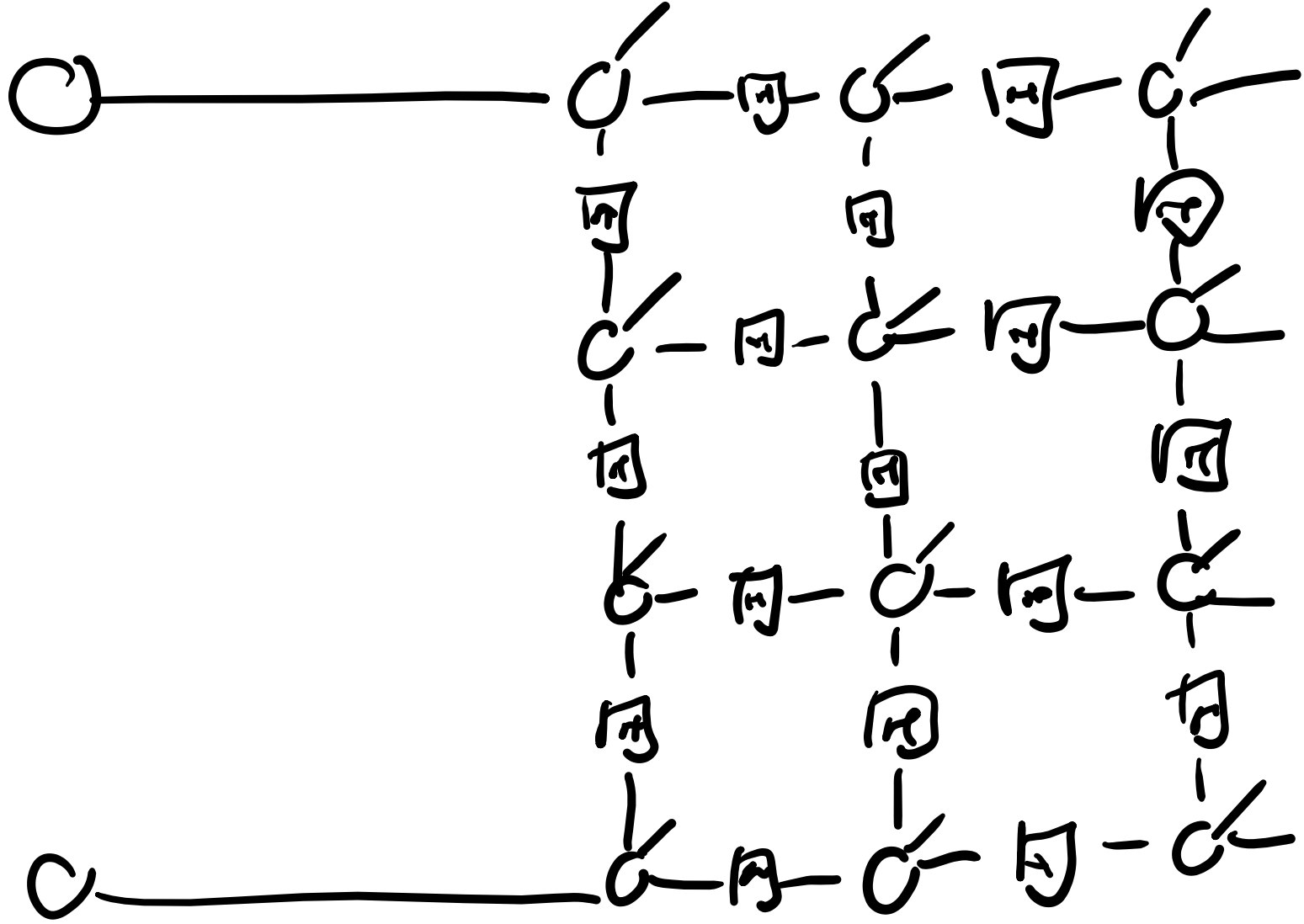


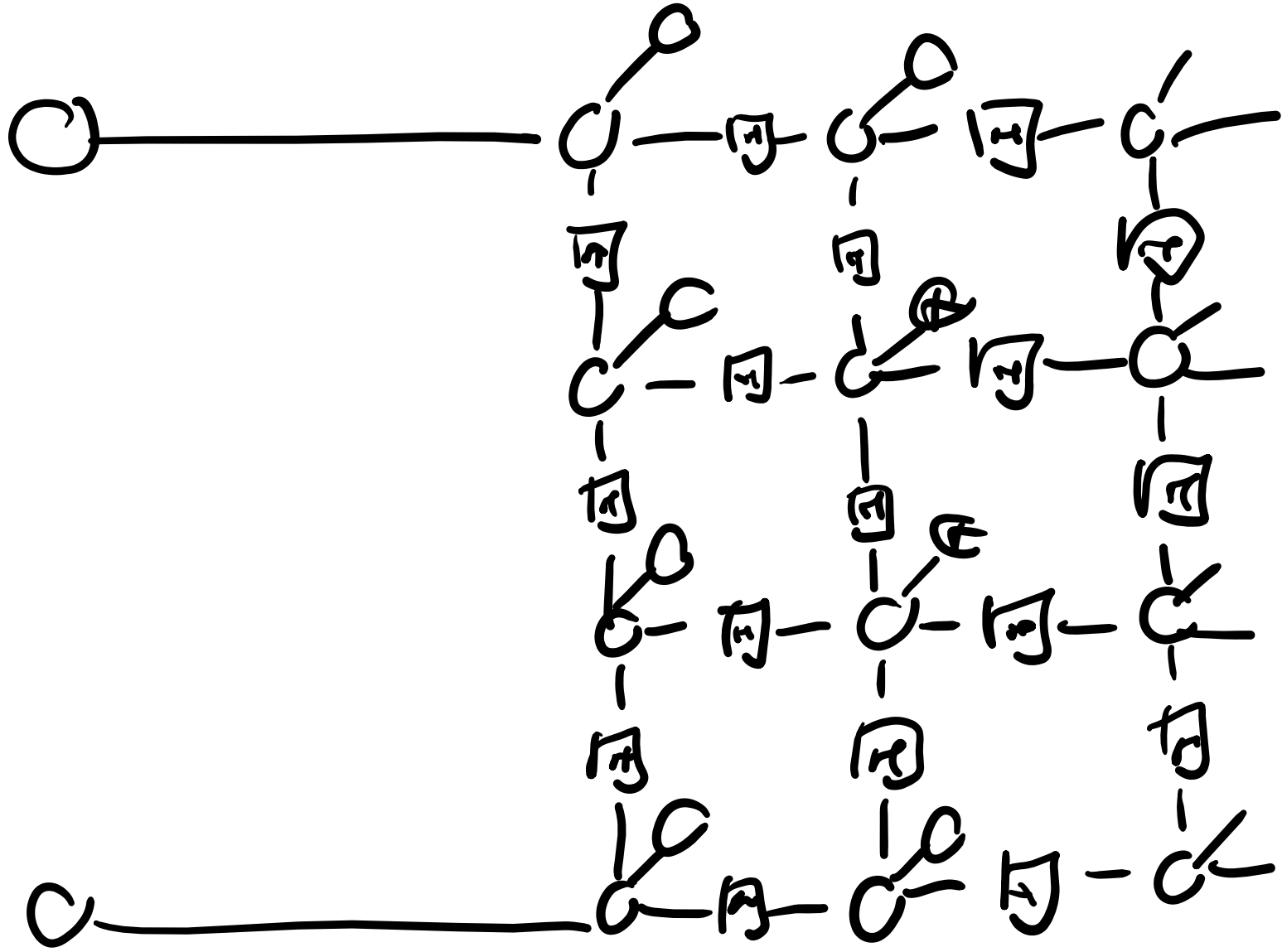


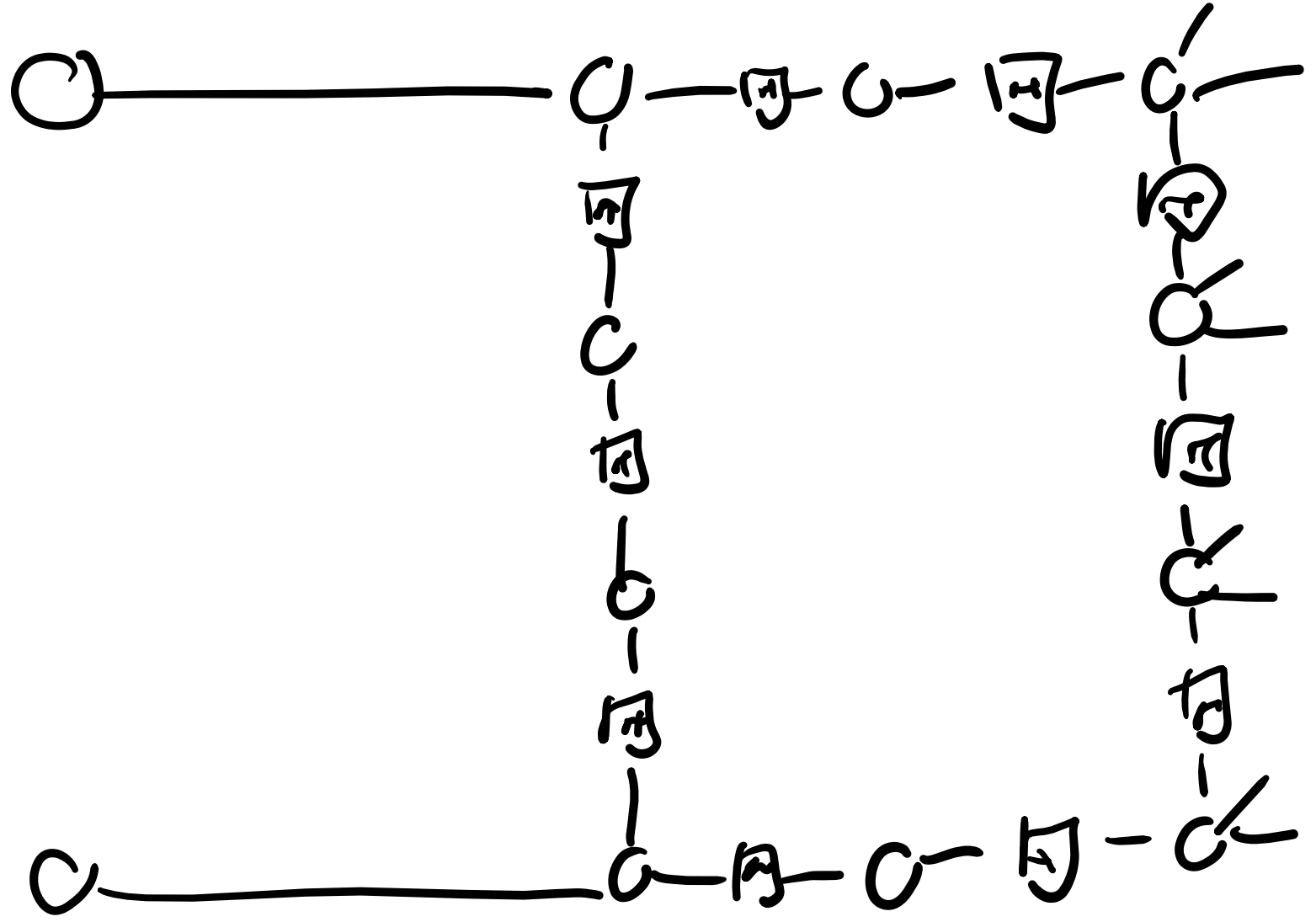


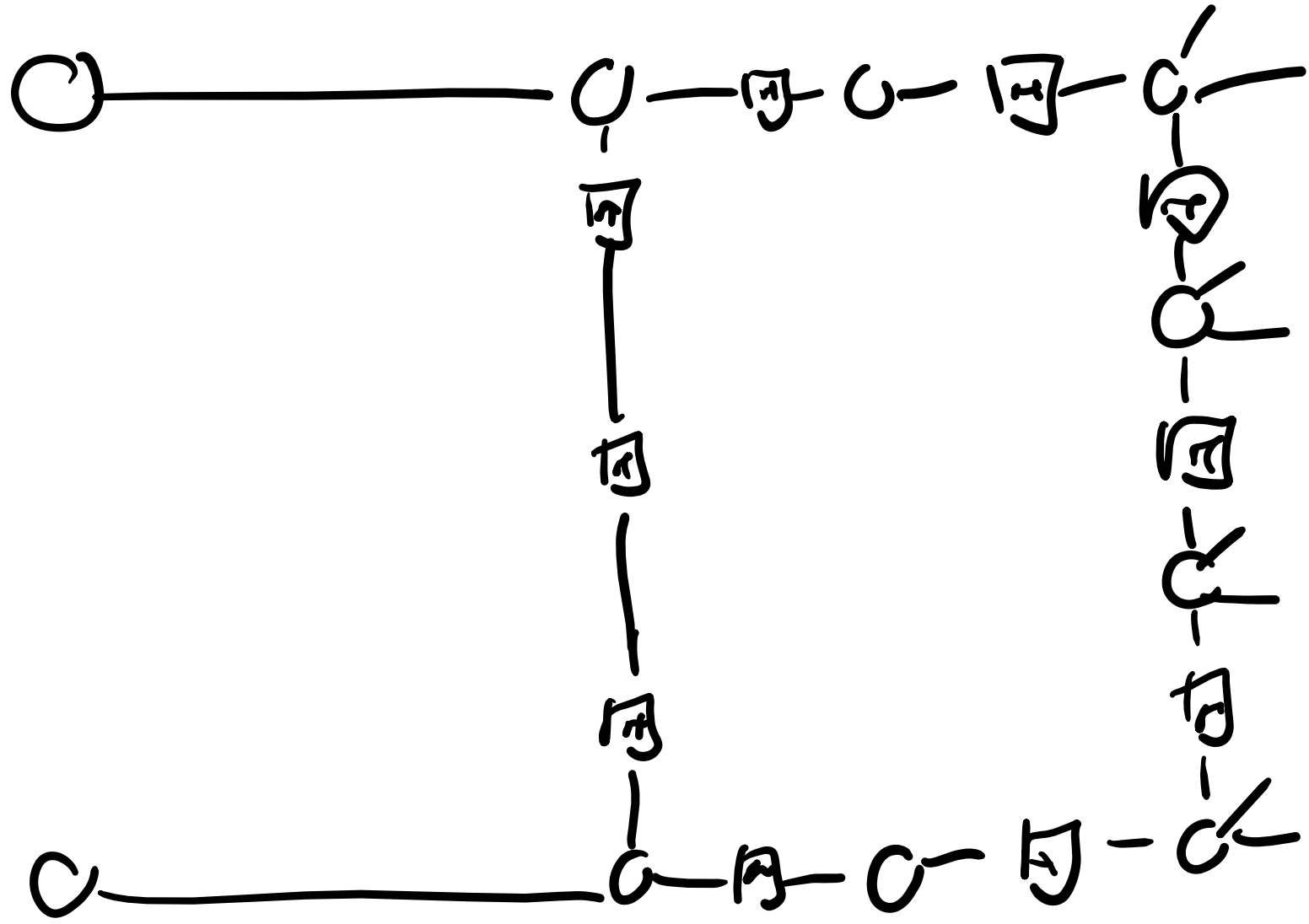


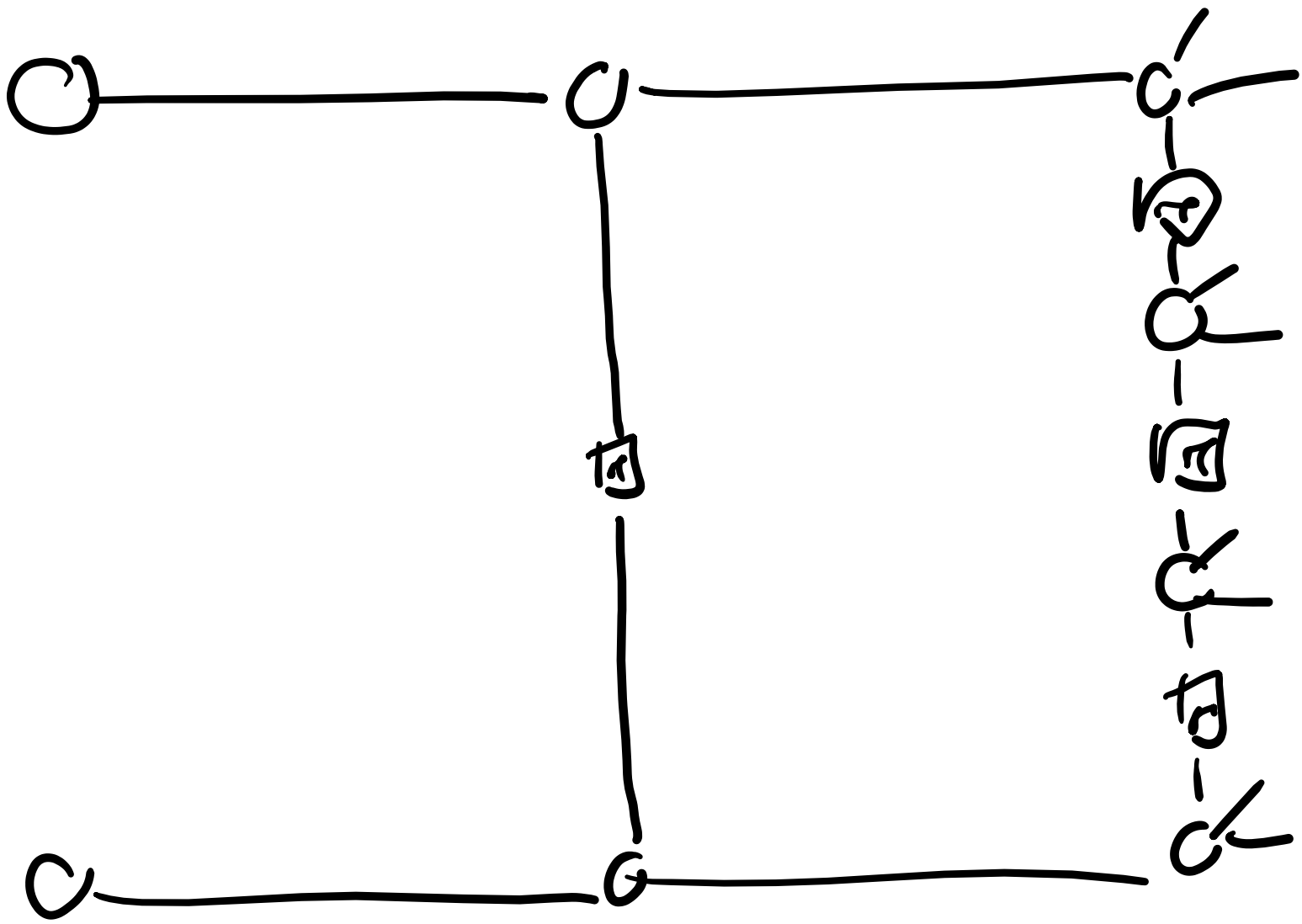


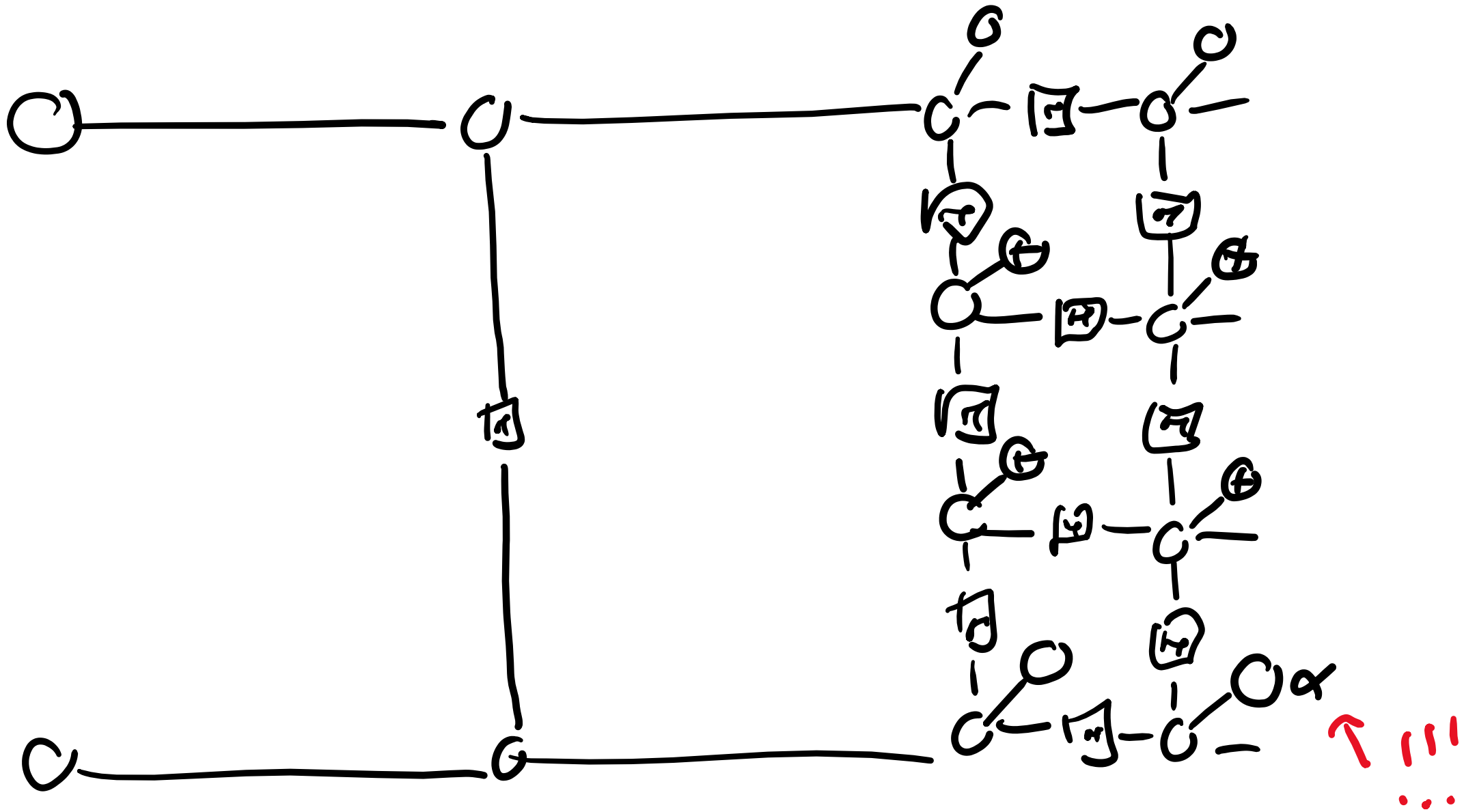


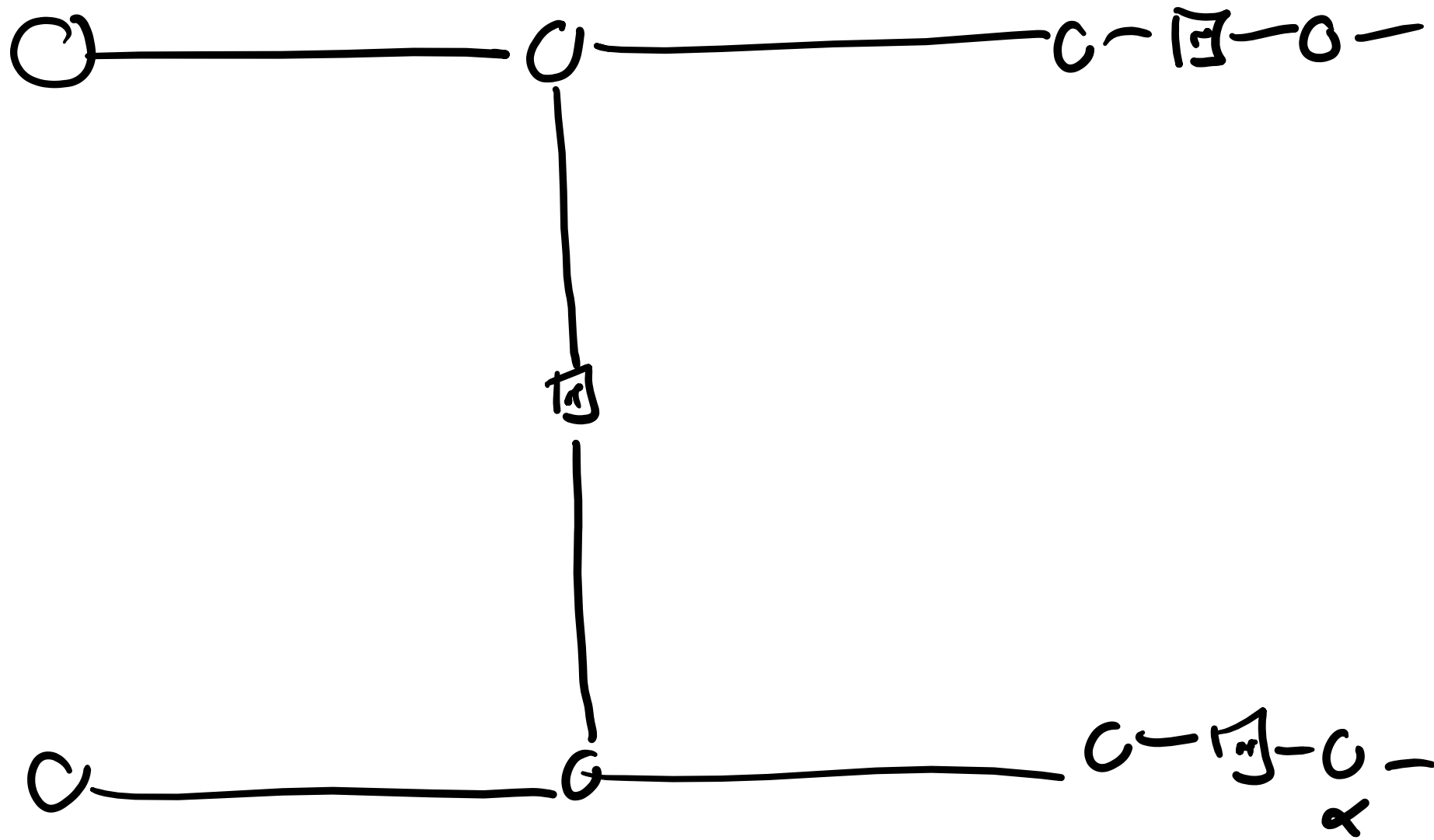


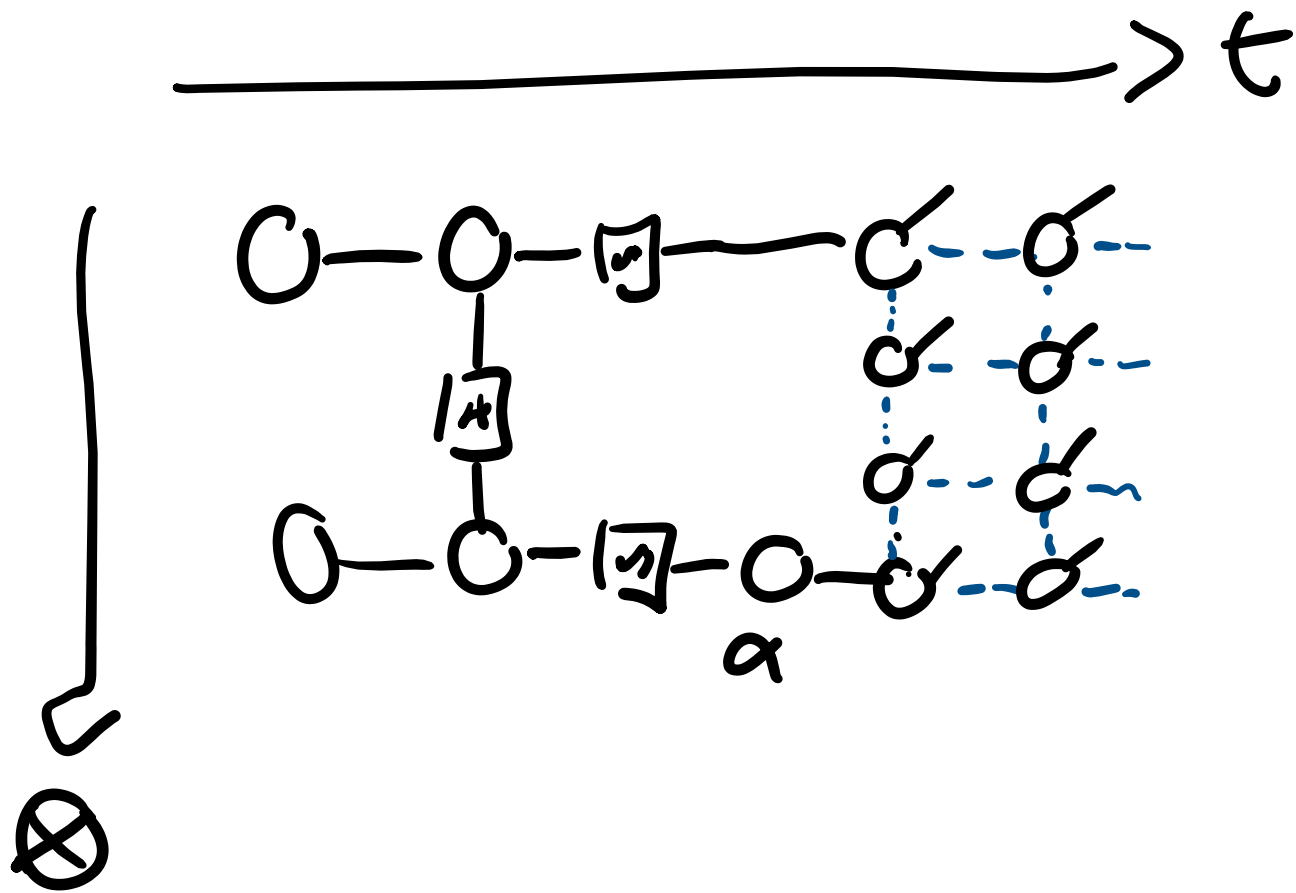












Cluster-state Γ BQC

<https://arxiv.org/pdf/2303.08829.pdf>