

# A Quantum Abacus

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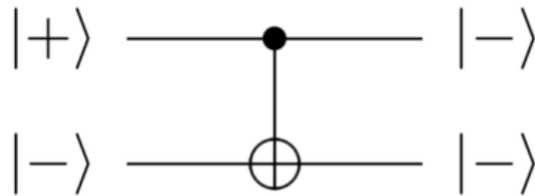
December 20, 2022

## Abstract

We explore a simple rewriting system<sup>1</sup> equivalent with the linear algebra normally associated with quantum gates with the goal of communicating an intuition to the beginner student of quantum computation. We start by building the axioms of this system through examples and eventually work out the proof of a simple theorem for which a beginner has no other resort than brute force (namely blind matrix multiplication).

## 1 Introduction

From the outset we want to state the theorem we want to prove:



The behavior of this gate is (at first) a bit counterintuitive: the control qubit changes while the target stays the same. By simple matrix multiplication one can verify the truth of this diagram.

## 2 Rules of Engagement

We start by defining  $|0\rangle$  as follows:

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \bigcirc$$

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<sup>1</sup>The system presented here has been introduced by Terry Rudolph in 2017.

Likewise we define

$$|1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \bullet$$

Now we introduce the X gate defined by:  $\sigma(\circ) = \bullet$  and  $\sigma(\bullet) = \circ$ . The reader can easily verify that this definition is a special case of the traditional definition (for adequately chosen values of  $\alpha$  and  $\beta$ ):

$$\sigma(\alpha|0\rangle + \beta|1\rangle) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \beta \\ \alpha \end{pmatrix}$$

The X gate (denoted by  $\sigma$  here) is also known as the quantum NOT gate. It is a one-qubit gate. It is easy to check that  $\sigma(\sigma(\circ)) = \circ$  and  $\sigma(\sigma(\bullet)) = \bullet$  in other words  $\sigma^2 = \mathbb{I}_2$  (the unit matrix of size 2). Here's another one-qubit gate, the Hadamard gate:

$$\Xi = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

The Hadamard gate acts as follows:

$$\Xi|0\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}} = |+\rangle$$

We write this as follows in our notation:

$$\Xi(\circ) = \{\circ, \bullet\}$$

For the other input the behavior is:

$$\Xi|1\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}} = |-\rangle$$

In our notation this is:

$$\Xi(\bullet) = \{\circ, \overline{\bullet}\}$$

Now since  $\Xi(\Xi(\circ)) = \circ$  and  $\Xi(\Xi(\bullet)) = \bullet$  (the Hadamard gate does share this property<sup>2</sup> with the X gate) the question then becomes:

- what should the axioms of behavior be for the new  $\{\}, \}$  operator?

We start by asking for the operator to be linear:

$$\Xi(\Xi(\circ)) = \Xi(\{\circ, \bullet\}) = \{\Xi(\circ), \Xi(\bullet)\} = \{\{\circ, \bullet\}, \{\circ, \overline{\bullet}\}\}$$

From this we conclude that

$$\{\{\circ, \bullet\}, \{\circ, \overline{\bullet}\}\} = \{\circ, \bullet, \circ, \overline{\bullet}\} = \circ$$

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<sup>2</sup>The reader can readily check this via matrix multiplication.

Likewise the following must hold:

$$\Xi(\Xi(\bullet)) = \Xi(\{\circ, \overline{\bullet}\}) = \{\Xi(\circ), \Xi(\overline{\bullet})\} = \{\{\circ, \bullet\}, \overline{\Xi(\bullet)}\}$$

Now to further pin down the behavior of this new operator we must further propagate the constraints from the most recent equality into its axiomatic rules of engagement. Thus,

$$\{\{\circ, \bullet\}, \overline{\Xi(\bullet)}\} = \{\{\circ, \bullet\}, \overline{\{\circ, \bullet\}}\} = \bullet$$

For that to happen we need to have:

$$\overline{\{\circ, \bullet\}} = \{\overline{\circ}, \overline{\bullet}\} = \{\overline{\circ}, \bullet\}$$

Then:

$$\{\{\circ, \bullet\}, \overline{\{\circ, \bullet\}}\} = \{\circ, \bullet, \overline{\circ}, \bullet\} = \{\circ, \bullet, \overline{\circ}, \bullet\} = \bullet$$

### 3 Rules of Entanglement

Now to study entanglement<sup>3</sup> we need two-qubit gates.

With two qubits the order matters:

$$|01\rangle = |0\rangle \otimes |1\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \circ\bullet \neq \bullet\circ$$

It is traditional to introduce the SWAP ( $\curvearrowright$ ) gate first, for example:

$$\curvearrowright |10\rangle = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = |01\rangle$$

In our notation we have:

$$\curvearrowright (\bullet\circ) = \circ\bullet$$

We now introduce the C-NOT ( $\leftarrow\mathcal{P}$ ) gate, through an example:

$$\leftarrow\mathcal{P} |10\rangle = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = |11\rangle$$

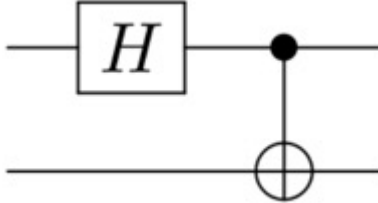
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<sup>3</sup>Note, however, that studying (or representing) entanglement is not our main goal here. Likewise, while this paper is not a tutorial, it does outline the how (and the why) of a tutorial.

In our notation that becomes:

$$\leftrightarrow_P (\bullet \circ) = \bullet \bullet$$

C-NOT distinguishes between its two arguments as follows: it first requires the control qubit, and then the target qubit. Let's look at a simple circuit now:



This is the circuit that creates the Bell states, as a function of its two inputs.  
For example:

$$\leftrightarrow_P (\Xi(\circ)\circ) = |\Phi^+\rangle = \{\circ\circ, \bullet\bullet\}$$

Let's work this out:

$$\leftrightarrow_P (\Xi(\circ)\circ) = \leftrightarrow_P (\{\circ, \bullet\}\circ) = \{\leftrightarrow_P (\circ\circ), \leftrightarrow_P (\bullet\circ)\} = \{\circ\circ, \bullet\bullet\}$$

Now, as an exercise, let's try to answer each of the following two questions:

1. Is this quantum state an entangled state?

$$\Psi = \sqrt{\frac{1}{4}} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

Answer: no, because  $\{\circ\circ, \circ\bullet, \bullet\circ, \bullet\bullet\} = \{\circ, \bullet\}\{\circ, \bullet\}$

2. Is this quantum state an entangled state?

$$\Psi = \frac{|00\rangle + |01\rangle}{\sqrt{2}}$$

Answer: no, because  $\{\circ\circ, \circ\bullet\} = \{\circ\}\{\circ, \bullet\} = \circ \{\circ, \bullet\}$

In both cases we invoke distributivity.

## 4 Conclusion

It's time to prove our theorem:

$$\leftarrow^P (|+-\rangle) = |--\rangle$$

We start by reminding ourselves that  $|+\rangle = \{\circ, \bullet\}$  and  $|-\rangle = \{\circ, \bar{\bullet}\}$

So now we need to calculate:

$$\begin{aligned} \leftarrow^P (|+-\rangle) &= \leftarrow^P (\{\circ, \bullet\} \{\circ, \bar{\bullet}\}) \\ &= \leftarrow^P (\{\circ\circ, \circ\bar{\bullet}, \bullet\circ, \bullet\bar{\bullet}\}) \\ &= \{\leftarrow^P (\circ\circ), \leftarrow^P (\circ\bar{\bullet}), \leftarrow^P (\bullet\circ), \leftarrow^P (\bullet\bar{\bullet})\} \\ &= \{\circ\circ, \circ\bar{\bullet}, \bullet\bar{\bullet}, \bullet\bar{\circ}\} \\ &= \{\circ\circ, \circ\bar{\bullet}, \bar{\bullet}\bar{\bullet}, \bar{\bullet}\bar{\circ}\} \\ &= \{\circ\circ, \circ\bar{\bullet}, \bar{\bullet}\circ, \bar{\bullet}\bar{\bullet}\} \\ &= \{\circ\{\circ, \bar{\bullet}\}, \bar{\bullet}\{\circ, \bar{\bullet}\}\} \\ &= \{\circ, \bar{\bullet}\} \{\circ, \bar{\bullet}\} \\ &= |--\rangle \end{aligned}$$

This concludes our tutorial example. We have demonstrated all aspects of the rewriting system and have shown its applicability. In future work we will take a more formal approach and better describe why a beginner might find this algebraic approach more accessible than the traditional approach.

## References

Besides the book by Terry Rudolph we also refer the reader to a video by Gerardo Ortiz available on the IU QSEc website. Work described herein is unpublished, still under development. While we believe everything stated here to be correct, some things may not have been explained in full detail. Please let us know if you see any typos or if you think anything is inaccurate, and should you have any questions, of any kind, just please let us know.

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