Definition. Let $f, g\colon \mathbb{N} \to \mathbb{R}^+$. We say that $f(n)$ *is of order* $g(n)$, written $f(n) \in O(g)(n)$, if there exist $N \in \mathbb{N}$ and $C \in \mathbb{R}$ such that for all $n \leq N$, $f(n) \leq C \cdot g(n)$.

$$\exists C \in \mathbb{R}, N \in \mathbb{N}\colon\ [\forall n \leq N\colon f(n) \leq C \cdot g(n)]$$

To show that $f(n) \in O(g)(n)$, one typically finds appropriate values (existential witnesses) for N and C.

Example 1. $2n^2 + 5n + 3 \in O(n^2)$.

Find a *dominating value* in terms of n^2 for each term in $2n^2 + 5n + 3$.

$$\begin{array}{ccccc} 2n^2 & + & 5n & + & 3 \\ \& & & \& & & \& \\ 0 \leq n & & 5 \leq n & & 2 \leq n \\ \Downarrow & & \Downarrow & & \Downarrow \\ 2n^2 \leq 2n^2 & & 5n \leq n^2 & & 3 \leq n^2 \\ \text{(a)} & & \text{(b)} & & \text{(c)} \end{array}$$

To satisfy (a–c), take $N = 5$. To satisfy (a+b+c), take $C = 4$.

$$\forall n \geq 5\colon \underbrace{2n^2 + 5n + 3}_{f(n)} \leq 4 \underbrace{n^2}_{g(n)}$$

Example 2. $2^{(n+10)} \in O(2^n)$.

Write down the inequality and "solve" it.

$$\begin{array}{rcll} 2^{(n+10)} & \leq & C \cdot 2^n & n \geq N,\ C > 0 \\ 2^{10} \cdot 2^n & \leq & C \cdot 2^n & \text{since } 2^{(n+10)} = (2^n)(2^{10}) \\ C \cdot 2^n & \leq & C \cdot 2^n & \text{if let } C = 2^{10} \end{array}$$

So this claim is true if $C > 0$. Since both sides of this inequality are now the same, the inequality is true for any value of n. We can take $N = 0$, then. So:

$$\forall n \geq 0\colon 2^{n+10} \leq 1024 \cdot 2^n$$

That is,

$$2^{(n+10)} \in O(2^n) \text{ for } C = 2^{10} \text{ and } N = 0$$

Example 2. $n^2 \notin O(n)$. Assume C and N exist such that, for all $n \geq N$, $n^2 \leq C \cdot n$.. Since $0 \leq n$ (considered as a real number), we can divide both sides of the inequation by n to get $n \leq C$. However, this inequality does not hold whenever $n > C$, a contradiction to our assumption. In particular, no such C exists and, therefore, $n^2) \notin O(n)$.

□

Example 3. $n(\log_2 n \notin O(n)$.

Assume $n(\log_2 n) \in O(n)$ and deduce a contradiction.

Assume C and N exist for which

$$\begin{array}{rcll} n(\log_2 n) & \leq & C \cdot n & \underline{\text{for all}}\ n \geq N \\ & \Downarrow & & \\ \log n & \leq & C & (\text{multiply by } \frac{1}{n},\ n \geq N \geq 0) \\ n & \overset{?!}{\leq} & 2^C & (\text{raise both sides to a power of 2, } n \geq 0) \\ & & & \underline{\text{for all}}\ n \geq N\ ?! \end{array}$$

The last inequality doesn't hold for arbitrarily large n, so we have reached a contradiction. The original assumption that $n(\log_2 n) \in O(n)$ must be false, and this is just what we set out to demonstrate.

Two explain the proof in a more "linear" way, one could write...

Assume for the purpose of contradiction that N and C exist such that, for all $n \geq N$, $n(\log_2 n) \leq C \cdot n$. Now consider any n greater than the larger of N and 2^C. Then since $\log_2$ is an increasing function,

$$n(\log_2 n) > n(\log_2 2^C) = n \cdot C.$$

This contradicts our assumption, so either N or C do not exist. Therefore $n(\log_2 n) \notin O(n)$.

□

Example 4. $n^2 + 7) \notin O(3n + 5)$.

Again, prove this by contradiction. Keep in mind that we are seeking a "big enough n" to refute the assumption, "For some N and C, $n^2 + 7 \leq C \cdot (3n + 5)$."

(a) Assume $n^2 + 7 \in O(3n + 5)$ and let N and C be the existential witnesses.

(b) Then for $n \geq N$, $n^2 + 7 \leq C(3n + 5) = 3Cn + 5C$.

(c) Subtracting 7 from both sides, $n^2 \leq 3Cn + 5C - 7$.

(d) Divide both sides by n to get $n \leq 3C + \frac{5C}{n} - \frac{7}{n}$.

(e) If $n > 5C$ and $n > 7$ we can replace both fractions by 1 and still preserve the inequality, $n \leq 3C + 2$.

(f) Thus, if we take n to be the larger of N and $3C + 2$, the inequality cannot hold.

This argument demonstrates by contradiction that C and N do not exist, proving the result.

□

Proposition. If $f(n) \in Og(n)$ and $g(n) \in Oh(n)$ then $f(n) \in Oh(n)$.

PROOF: Assume N_1 and C_1 are witnesses to $f(n) \in Og(n)$ and N_2 and C_2 are witnesses to $g(n) \in Oh(n)$. Let N_3 be the greater of N_1 and N_2. For all $n \geq N_3$

$$\begin{array}{rcll} f(n) & \leq & C_1 \cdot (g(n)) & (n \geq N_1)\text{and } f(n) \in O(g(n)) \\ & \leq & C_1 \cdot (C_2(n)) & (n \geq N_2 \text{ and } g(n) \in O(h(n))) \end{array}$$

Thus, for all $n \geq N_3$, and for $C_3 = C_1 \cdot C_2$, $f(n) \leq C_3 \cdot h(h)$. That is, $f(n) \in O(h(n))$ with witnesses N_3 and C_3.

□

Exponential Order

[Taken from lecture notes by Danial Leivant, 2005]

Proposition. For all $n \in \mathbb{N}$, $n < 2^n$.

PROOF: by induction on $n \in \mathbb{N}$. BASE CASE: $0 < 1 = 2^0$.

INDUCTION: Assume $k < 2^k$. Then $k+1 \overset{H}{<} 2^k+1 < 2^k+2^k = 2 \cdot 2^k = 2^{k+1}$.

□

Proposition. For all $n \in \mathbb{N}$, $n \geq 4$ implies $n^2 < 2^n$.

PROOF: We proceed by induction. BASE CASE: $4^2 \leq 16 = 2^4$.

INDUCTION: Assume that $k^2 \leq 2^k$. Then

$$\begin{array}{rcll} (x+1)^2 & = & x^2 + 2x + 1 & \\ & \overset{\text{H}}{<} & 2^k + 2x + 1 & \text{I.H.} \\ & < & 2^k + 2^k & \text{by the Lemma above} \\ & = & 2^{k+1} & \end{array}$$

□

Proposition. For all $n \geq 3$, $n^2 > 2n + 1$.

PROOF: by induction on $n \geq 3$. BASE CASE: $3^2 = 9 > 7 = 2 \cdot 3 + 1$. INDUCTION:

$$\begin{array}{rcll} (x+1)^2 & = & x^2 + 2x + 1 & \\ & \overset{\text{H}}{=} & (2x+1) + 2x + 1 & \text{I.H., } n \geq 4 \\ & = & 4x + 2 & \\ & > & 2x + 3 & \text{since } x > 1 \\ & = & 2(x+1) + 1 & \end{array}$$

□

sn't right?!

Corollary. If $C \geq 4$ then for all $n \geq 4$, $C \cdot n < 2^n$.

Discussion. We are going to show that $n^k \in O[2^n]$ *provided* that n is a power of 2, that is $m = 2^m$. Thus, means we are only "sampling" the functions for values of $n \in \{1, 2, 4, 8, 16, \ldots\}$ to see which dominates the other. That this is good enough is not obvious. but is a consequence of the fact that, considered as functions over $\mathbb{R}$, they increase "smoothly."

Proposition. For all $k \geq 4$ there exists an $m \in \mathbb{N}$ such that $2^{2^m} > (2^m)^k$.

Proof: Note that the right-hand side, $(2^m)^k = 2^{mk}$. By the Corollary above, since $k \geq 4$, we have $2^m > km$ for all $m \geq k$. raising 2 to the power of both sides preserves the inequality, yielding $2^{2^m} > 2^{km} = (2^m)^k$, as desired. □

□

Corollary. For every $k \geq 1$ and every $C \in \mathbb{R}$, there exists an $m \in \mathbb{N}$ for which $2^{2^m} < C \cdot (2^m)^k$.

Proof: By the Proposition above there is a constant D for which $(2^m)^k < 2^{2^m}$ whenever $2^m > D$. Taking m such that 2^m is the greater of C and D we have

$$C \cdot (2^m)^k \leq (2^m)(2^m)^k = (w^m)^{k+1} < 2^{2^m}$$

. □