C241 Homework Assignment 9

1.

The language L and functions R, A, and T defined below are the same as in Section 7.6.

	$L \subseteq \{a,\ b,\ \bullet\}^+$
1.	$\bullet \in L$
2a.	$u \in L \Rightarrow au \in L$
2b.	$u \in L \Rightarrow bu \in L$
3.	*n. e.*

	$A\colon L^2 \to L$	$R\colon L \to L$	$T\colon L^2 \to L$
1.	$A(\bullet, v) = v$	$R(\bullet) = \bullet$	$T(\bullet, v) = v$
2a.	$A(au, v) = aA(u, v)$	$R(au) = A(R(u), a\bullet)$	$T(au, v) = T(u, av)$
2b.	$A(bu, v) = bA(u, v)$	$R(bu) = A(R(u), b\bullet)$	$T(bu, v) = T(u, bv)$

It is proved in the book that

- **Proposition 7.6.** *A is associative; that is, for all* $u, v, w \in L$, $A(u, A(v, w)) = A(A(u, v), w)$.
- **Proposition 7.9.** *Assuming Proposition 7.8, below, R is self-cancelling; that is, for all* $u \in L$, $R(R(u)) = u$.

Prove the following:

(a) **Proposition 7.7.** *For all* $u \in L$, $A(u, \bullet) = u$.

(b) **Proposition 7.8.** *For all* $u, v \in L$, $R(A(u, v) = A(R(v), R(u))$.

(c) **Proposition 7.11.** *For all* $u, v \in L$, $T(u, v) = A((R(u), v)$.

(d) **Proposition 7.10.** *For all* $u \in L$, $T(u, \bullet) = R(u)$.

(e) **Proposition 7.12.** *For all* $u \in L$, $T(T(u, \bullet), \bullet) = u$.

2. The program below is called *Wensley's algorithm* for computing *the quotient of real numbers x and y to within tolerance t. Use the Theorem on Loop Invariants to prove the this program satisfies the post-condition* $\{z \le x/y < z+t\}$.

$\{0 \le x < y \le 1\}$
begin
z := 0 ; d := 1 ; u := 0 ; v := $\frac{1}{2}y$;
while $d > t$ **do**
  $\{\ \textsc{Inv} \equiv z \le x/y < z+d \wedge u = zy \wedge v = \frac{1}{2}dy\}$
  begin
  d := $\frac{1}{2}d$;
  if $u+v > x$ **then skip**
  else begin z := $z+d$; u := $u+v$; **end** ;
  v := $\frac{1}{2}v$
  end
end
$\{z \le x/y < z+t\}$

3. Define $F\colon \mathbb{N} \to \mathbb{N}$ and $G\colon \mathbb{N}^2 \to \mathbb{N}$ as follows:

$$\begin{aligned} F(0) &= 1 & \qquad G(0, m) &= m \\ F(k+1) &= (k+1) \times F(k) & \qquad G(k+1, m) &= G(k,\ m \times (k+1)) \end{aligned}$$

(a) Prove by induction on $n \in \mathbb{N}$: For all $n, m \in \mathbb{N}$, $G(n, m) = m \times G(n, 1)$.

(b) Prove: For all $n \in \mathbb{N}$, $F(n) = G(n, 1)$.

4. Performance estimation for recursive programs often involves *recurrence relations* like the one below. Let $a \in \mathbb{N}$. The function $T\colon \mathbb{N} \to \mathbb{N}$ is defined recursively by

$$\begin{aligned} T(0) &= a \\ T(k+1) &= T(k) + k + 1 \end{aligned}$$

We would like to find a *closed form* for T, that is, and an algebraic expression that does not involve recursion

Prove that for all $n \in \mathbb{N}$, $T(n) = a + \dfrac{n^2 + n}{2}$.

Supplemental Problem. *H241 students should attempt this programming problem, but don't spend more than two or three hours on it.*

In a programming language of your choice, write a program that takes no input and outputs its own source code.