C241 Homework Assignment 8

1.

Estimate the performance of the *Bubble Sort* program

```
begin
for i from 1 to N − 1 by 1 do
   begin
   for j from 1 to i − 1 by 1 do
     if A[j] ≤ A[j + 1]
     then skip;
     else
       begin
       t := A[j];
       A[j] := A[j + 1];
       A[j + 1] := t
       end
   end
end
```

2. Define two functions, f and g, for which $f \notin O(g)$ and $g \notin O(f)$.

3. Prove: *If* $f_1 \in O(g_1)$ *and* $f_2 \in O(g_2)$ *then* $f_1(n) \times f_2(n) \in O\big(g_1(n) \times g_2(n)\big)$

4. Is $2^n \in O(n!)$ or is $n! \in O(2^n)$?

5. Prove: *If* $f(n) \in O(h(n))$ *and* $g(n) \in O(h(n))$ *then* $f(n) + g(n) \in O(h(n))$.

6. Consider the program to the right, which computes over values in $\mathbb{N}$. Define the outer loop's invariant assertion I_1 to be:

$$I_1 \equiv z \cdot x^y = A^B$$

and the inner loop's invariant assertion I_2 to be:

$$I_2 \equiv z \cdot x^y = A^B \wedge y \neq 0$$

Since z is initialized to 1, I_1 is true when the program first reaches the outer loop; and since $I_2 \equiv I_1 \wedge y \neq 0$, I_2 holds whenever the inner loop is reached. Assuming that I_1 and I_2 are loop invariants, by the Theorem on Loop Invariants we will have $I_1 \wedge y = 0$ when the program terminates and hence

$$z \cdot x^y = z \cdot x^0 = z \cdot 1 = A^B$$

as desired. So it remains to be proved that I_1 and I_2 are invariants for their respective loops. Prove this.

HINT. *Start with the inner loop.*

```
{x = A ∧ y = B}
begin
z := 1;
while y ≠ 0 do {I1}
  begin
  while even?(y) do {I2}
    begin
    y := y/2
    x := x * x
    end;
  z := z * x;
  y := y − 1
  end
end
{z = A^B}
```

Supplemental Problem. (Hotel Paradox)
[*This problem appears in many forms in many places.*]

Three grad students go to a conference and decide to save money by sharing a hotel room. On checking in, the desk clerk says the charge for a room containing two beds and a cot is \$30. After the students have paid, the desk clerk realizes that the charge should have been only \$25. She gives the doorkeeper a \$5 bill and asks that it be refunded to the students.

On the way to the room the doorkeeper realizes that he cannot divide \$5 by 3 evenly, so he decides to refund the students \$1 each and keeps \$2 for himself.

The students are happy that they paid only \$9 each for the room. The doorkeeper is happy that he gets a \$2 "tip." But $3 \times \$9 + \$2 = \$29$. Where did the other \$1 go?