C241 Homework Assignment 7

1. Prove that for all whole numbers n,

$$\sum_{i=0}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

Solution

The proof is by induction on k with hypothesis

$$H(k) \sum_{i=1}^{k} i^2 = \frac{k(k+1)(2k+1)}{6}$$

BASE CASE: *To prove $H(1)$,*

$$\sum_{i=1}^{1} i^2 = 1 = \frac{6}{6} = \frac{1 \cdot 2 \cdot 3}{6} = 1 \cdot (1+1) \cdot (2 \cdot 1 + 1)6$$

INDUCTION STEP: *Assume $H(k)$. Now,*

$$\begin{aligned}
\sum_{i=0}^{k+1} i^2 &= \left(\sum_{i=0}^{k} i^2\right) + (k+1)^2 && \textit{(expand } \textstyle\sum\textit{)} \\
&= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 && \textit{(IH)} \\
&= \frac{k(k+1)(2k+1)}{6} + \frac{6(k+1)^2}{6} && \textit{(multiply by } 1 = 6/6\textit{)} \\
&= \frac{k(k+1)(2k+1) + 6(k+1)^2}{6} && \textit{(add fractions)} \\
&= \frac{(k+1)\big(k(2k+1) + 6(k+1)\big)}{6} && \textit{(factor } (k+1)\textit{)} \\
&= \frac{(k+1)(2k^2+7k+6)}{6} && \textit{(combine like terms)} \\
&= \frac{(k+1)(k+2)(2k+3)}{6} && \textit{(factor } 2k^2+7k+6\textit{)} \\
&= \frac{(k+1)((k+1)+1)(2(k+1)+1)}{6} && \textit{(arithmetic)}
\end{aligned}$$

This completes the induction case and the proof of the problem.

2. Prove that for all $n \in \mathbb{N}$,

$$\sum_{i=0}^{n} i(i!) = (n+1)! - 1$$

Solution

The proof is by induction on k with hypothesis

$$H(k) \equiv \sum_{i=0}^{k} i(i!) = (k+1)! - 1$$

Recall that

$$n! = \begin{cases} 1 & \textit{if } n = 0 \\ 1 \cdot 2 \cdot \cdots \cdot n & \textit{otherwise} \end{cases}$$

BASE CASE:

$$\sum_{i=1}^{1} i(i!) = 1(1!) = 1 = 2 - 1 = 2! - 1 = (2+1)! - 1$$

INDUCTION STEP: *Assume* $H[k] \equiv \sum_{i=0}^{k} i(i!) = (k+1)! - 1$. *Now,*

$$\begin{aligned}
\sum_{i=0}^{k+1} i(i!) &= \left(\sum_{i=0}^{k} i(i!)\right) + (k+1)(k+1)! && \textit{(expand } \textstyle\sum\textit{)} \\
&= ((k+1)! - 1) + (k+1)(k+1)! && \textit{(induction hypothesis)} \\
&= ((k+1)! + (k+1)(k+1)!) - 1 && \textit{(rearrange sums)} \\
&= (1 + k + 1)(k+1)! - 1 && \textit{distribute } (k+1)! \\
&= (k+2)(k+1)! - 1 && \textit{(simplify)} \\
&= (k+2)! - 1 && \textit{(meaning of `!')} \\
&= ((k+1)+1)! - 1 && \textit{(as desired)}
\end{aligned}$$

This proves $H[k+1]$ and completes the induction.

3. Prove that for all whole numbers n, 6 evenly divides $n^3 - n$.

SOLUTION

The proof is by induction on $k \in \mathbb{N}$ *with hypothesis* $H[n] \equiv$ *"6 evenly devides* $n^3 - n$*"*

BASE CASE *(*$H[1]$*).* $1^3 - 1 = 1^3 - 1 = 0$*, which is evenly divisible by 6.*

INDUCTION *(*$H[k] \Rightarrow H[k+1]$*). Assume that* $k^3 - k$ *is evenly divisible by* 6. *Now,*

$$\begin{array}{lll} & (k+1)^3 - (k+1) & \\ = & (k^3 + 3k^2 + 3k + 1) - (k+1) & \textit{Expanding } (k+1)^3 \\ = & (k^3 + 3k^2 + 2k) & \textit{Combining like terms} \\ = & k(k^2 + 3k + 2) & \textit{Distributing } k \\ = & k(k+1)(k+2) & \textit{Factoring } k^2 + 3k + 2 \end{array}$$

By Proposition A (below) this number is divisible by 3. And since at least one of k*,* $k+1$*, or* $k+2$ *is an even number, it is also divisible by 2. That is,* $k(k+1)(k+2) = 2 \cdot 3 \cdot q$*, for some* q*, and is therefore divisible by 6.*

Proposition A. *For all* $n \in \mathbb{N}$*,* $n(n+1)(n+2)$ *is divisible by 3.*

PROOF: *by induction on* n *with hypothesis,* $H[k] \equiv k(k{+}1)(k{+}2) = 3p$ for some p.

BASE CASE: $0 \cdot 1 \cdot 2 = 0 = 3 \cdot 0$.

INDUCTION CASE: *Assume* $k(k+1)(k+2) = 3p$ *for some* p.

$$\begin{array}{rcll} (k+1)(k+2)(k+3) & = & k(k+1)(k+2) + 3(k+1)(k+2) & \textit{(distributing over } k+3\textit{)} \\ & \overset{\text{IH}}{=} & 3p + 3(k+1)(k+2) & \textit{(Induction Hypothesis)} \\ & = & 3\,[p + (k+1)(k+2)] & \textit{(distributing the 3)} \end{array}$$

COMMENT: *It is reasonable to say that this proposition is obvious and need not be proven.*

4. Prove that for all whole numbers $n > 4$, $2^n > n^2$.

SOLUTION

The proof is by induction on k with hypothesis $H[k] \equiv 2^k > k^2$.
BASE CASE: *To prove $H[5]$,*

$$2^5 = 32 > 25 = 5^2$$

INDUCTION STEP: *Assume $2^k > k^2$. Now,*

$$\begin{aligned} 2^{k+1} &= 2 \cdot 2^k \\ &> 2 \cdot k^2 && \textit{(induction hypothesis, used twice)} \\ &= k^2 + k^2 \\ &> k^2 + 2k + 1 && (k > 4 \overset{?}{\Rightarrow} k^2 > 2k + 1) \\ &= (k+1)^2 \end{aligned}$$

The induction is finished once it is established that for any natural number $n > 4$, $n^2 > 2n + 1$. This fact is proved in the lemma below.

LEMMA *For all $n > 3$, $n^2 > 2n + 1$.*
PROOF: *The proof is by induction on k with hypothesis $H[k] \equiv k > 3,\ k^2 > 2k + 1$.*
BASE CASE:

$$3^2 = 9 > 7 = 3 \cdot 3 + 1$$

INDUCTION STEP: *Assume $k^2 > 2k + 1, k > 3$. Then*

$$\begin{aligned} (k+1)^2 &= k^2 + 2k + 1 \\ &> (2k+1) + 2k + 1 && \textit{(induction hypothesis)} \\ &> (2k+1) + 1 + 1 && (k > 1) \\ &= (2k+2) + 1 && \textit{(arithmetic)} \\ &= 2(k+1) + 1 && \textit{(arithmetic)} \end{aligned}$$

This completes the induction and the proof of the lemma.
The proof of the lemma completes the proof of the problem.

5. Use induction to prove that the sum of the first n odd numbers is equal to n^2.

That is, show: *For all* $n \in Nat$, $n > 0$, $\sum_{i=1}^{n}(2i-1) = n^2$.

SOLUTION

BASE CASE. $\sum_{i=1}^{1}(2i-1) = 2{\cdot}1 - 1 = 1 = 1^2$

INDUCTION. *Assume that* $\sum_{i=1}^{k}(2i-1) = k^2$.

$$\begin{aligned}
\sum_{i=1}^{k+1}(2i-1) &= 2(k+1) - 1 + \boxed{\sum_{i=1}^{k}(2i-1)} && \textit{(expanding } \textstyle\sum \textit{ by one term)} \\
&\overset{H}{=} 2(k+1) - 1 + \boxed{k^2} && \textit{(I.H.)} \\
&= k^2 + 2k + 1 && \textit{(simplifying)} \\
&= (k+1)^2 && \textit{(factoring)}
\end{aligned}$$

6. Consider the program

```
P:  begin
    {A, B > 0}
    q := 0;
    r := A;
ℓ:  while r ≥ B do
       begin
       q := q + 1;
       r := r − B
       end;
    end {A = qB + r ∧ r < B}
```

Use Theorem 5.1Loop Invariantstheorem.5.1 and invariant assertion

$$I \equiv A = qB + r$$

to prove that this program computes the quotient and remainder of A and B.

Solution

According to Theorem 5.1Loop Invariantstheorem.5.1, it we can show that assertion I is true when $\mathcal{P}$ first reaches ℓ and that executing the loop body leaves I true, then when $\mathcal{P}$ ends, I will be true and the test "$r \geq B$" will be false.

Thus, there are three things to prove: Note: Blue comments need not be included because they are implied by the use of Theorem 5.1Loop Invariantstheorem.5.1

1. Initialization (Base Case):

$$\{A, B > 0\}\ q := 0\ \ ;\ \ r := A\ \ \{A = qB + r\}$$

Since we have just assigned 0 to q and A to r we have $qB + r = 0 \cdot B + A = A$.

2. Invariance (Induction Case):

$$\{A = qB + r \wedge r \geq B\}\ q := q + 1\ \ ;\ \ r := r - B\ \ \{A = qB + r\}$$

The loop body computes new values for program variables $q' = q + 1$ and $r' = r - B$ (valid because $r \geq B$). So we have

$$\begin{aligned} q'B + r' &= (q+1)B + (r - B) && \text{(substituting for } r' \text{ and } q'\text{)} \\ &= qB + B + r - B \\ &= qb + r && \text{(simplifying)} \\ &\overset{\text{IH}}{=} A && \text{(assumption)} \end{aligned}$$

as desired.

3. Termination: *It remains to show that $I \wedge (r < b)$ implies the postcondition, $\{(A = qB + r) \wedge (r < B)\}$. But in this case the two formulas are identical, so the implication is tautologically valid.*

Supplemental Problem. (Unit Square Puzzle) Solve the Problem below assuming the following two facts. Then see if you can prove Theorems A and B.

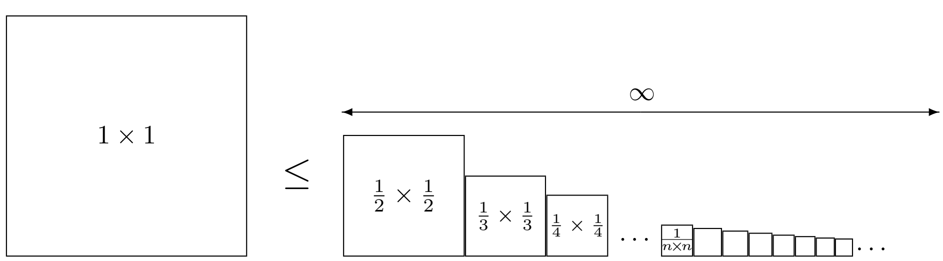
A. Theorem. *The* harmonic *sum,* $\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \cdots$ *diverges. That is,*

$$\lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k}\right) = \infty$$

B. Theorem. *The* geometric *sum* $1 + \frac{1}{2} + \frac{1}{9} + \frac{1}{16} + \cdots + \frac{1}{n^2} + \cdots$ *converges and is less than 2. That is,*

$$\lim_{n\to\infty}\left(\sum_{k=0}^{n}\frac{1}{k^2}\right) \leq 2$$

Suppose you have all the squares of size $\frac{1}{n} \times \frac{1}{n}$ for $n \in \mathbb{W}$. Since the area of the unit square (1×1) is 1, Theorem B suggests that it should be possible to fit all the smaller squares inside it without overlapping. On the other hand, Theorem B suggests that this might not be possible because, if you place all these squares next to each other, the resulting linear arrangement is infinitely long.



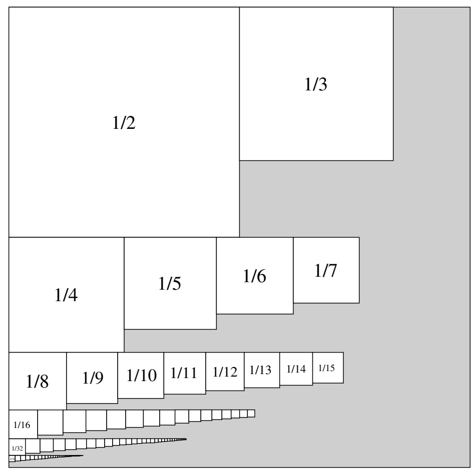
Problem. Determine whether and how all the $\frac{1}{n} \times \frac{1}{n}$ squares, $n > 1$, can be placed inside the unit square. If this problem has a solution, you should present a construction[1] showing how to do it, and prove any arithmetic facts needed to justify the correctness of your construction.

Solution

There is room to spare. Start by placing the $\frac{1}{2} \times \frac{1}{2}$ *square in the upper-left corner. Now placing the* $\frac{1}{3} \times \frac{1}{3}$ *square next to it. Now place the* $\frac{1}{4} \times \frac{1}{4}$, $\frac{1}{5} \times \frac{1}{5}$, $\ldots$, $\frac{1}{6} \times \frac{1}{6}$, *and*$\frac{1}{7} \times \frac{1}{7}$ *squares just below the first row.*

[1]Because we are dealing with infinitely many squares, your construction may not be an algorithm because it does not terminate. Even so, you must establish that each step of the procedure is well determined and possible.

Just below row $k-1$ add all the squares with sides of $\frac{1}{2^k}$ *through* $\frac{1}{2^{k+1}-1}$.



There are two things to prove in order to verify this solution. Both proofs are easy inductions.

A. *There is room vertically for all the rows because for all $n \in \mathbb{W}$,*

$$\sum_{i=1}^{n} \frac{1}{2^i} = 1 - 2^n \quad \left(\textit{thus, } \lim_{n \to \infty} \sum_{i=1}^{n} \frac{1}{2^i} \leq 1 \right)$$

B. *There is room for horizontally for each row because for all $n \in \mathbb{W}$,*

$$\sum_{i=2^n}^{2^{n+1}-1} i < 1$$