# C241 Homework Assignment 7

**1.** Prove that for all whole numbers $n$,

$$\sum_{i=0}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

**2.** Prove that for all $n \in \mathbb{N}$,

$$\sum_{i=0}^{n} i(i!) = (n+1)! - 1$$

**3.** Prove that for all whole numbers $n$, 6 evenly divides $n^3 - n$.

**4.** Prove that for all whole numbers $n > 4$, $2^n > n^2$.

**5.** Use induction to prove that the sum of the first $n$ odd numbers is equal to $n^2$.

That is, show: *For all* $n \in Nat,\ n > 0,\ \sum_{i=1}^{n}(2i-1) = n^2$.

**6.** Consider the program

```
𝒫: begin
   {A, B > 0}
   q := 0;
   r := A;
ℓ: while r ≥ B do
      begin
      q := q + 1;
      r := r − B
      end;
   end {A = qB + r ∧ r < B}
```

Use Theorem 5.1 and invariant assertion

$$I \equiv A = qB + r$$

to prove that this program computes the quotient and remainder of $A$ and $B$.

Supplemental Problem. (Unit Square Puzzle) Solve the Problem below assuming the following two facts. Then see if you can prove Theorems A and B.

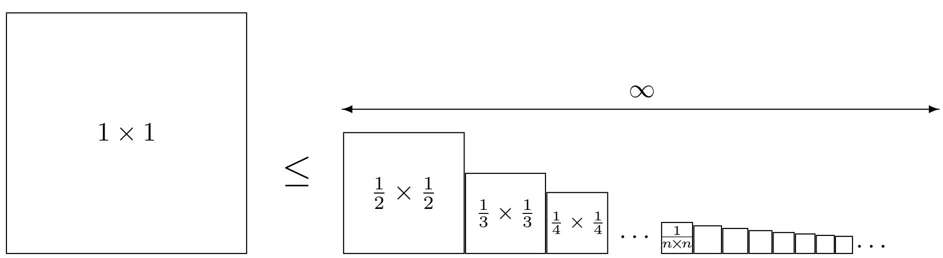
A. Theorem. *The* harmonic *sum,* $\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \cdots$ *diverges. That is,*

$$\lim_{n\to\infty} \left( \sum_{k=1}^{n} \frac{1}{k} \right) = \infty$$

B. Theorem. *The* geometric *sum* $1 + \frac{1}{2} + \frac{1}{9} + \frac{1}{16} + \cdots + \frac{1}{n^2} + \cdots$ *converges and is less than 2. That is,*

$$\lim_{n\to\infty} \left( \sum_{k=0}^{n} \frac{1}{k^2} \right) \leq 2$$

Suppose you have all the squares of size $\frac{1}{n} \times \frac{1}{n}$ for $n \in \mathbb{W}$. Since the area of the unit square $(1 \times 1)$ is 1, Theorem B suggests that it should be possible to fit all the smaller squares inside it without overlapping. On the other hand, Theorem B suggests that this might not be possible because, if you place all these squares next to each other, the resulting linear arrangement is infinitely long.




Problem. Determine whether and how all the $\frac{1}{n} \times \frac{1}{n}$ squares, $n > 1$, can be placed inside the unit square. If this problem has a solution, you should present a construction[1] showing how to do it, and prove any arithmetic facts needed to justify the correctness of your construction.

[1]Because we are dealing with infinitely many squares, your construction may not be an algorithm because it does not terminate. Even so, you must establish that each step of the procedure is well determined and possible.