C241 Homework Assignment 6

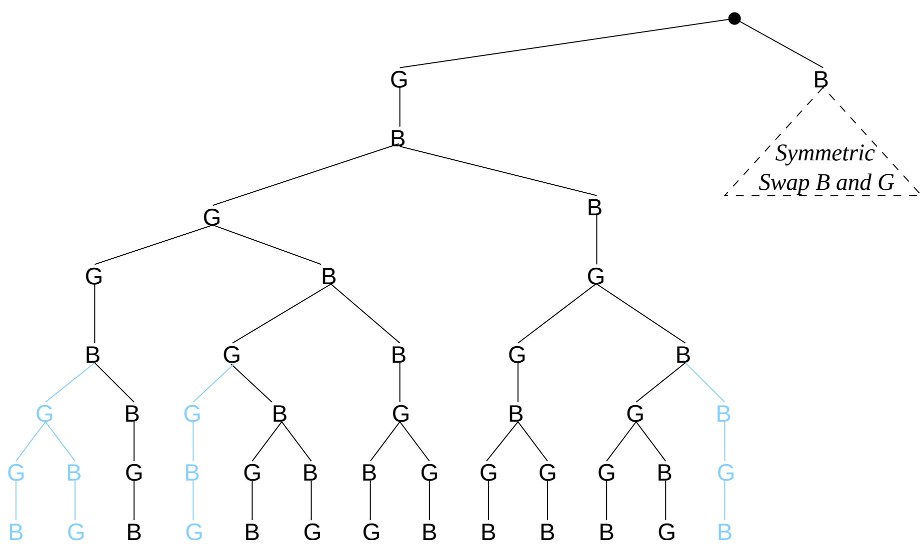
1. Suppose you want to assign seats for a single row of 4 guys and 4 gals in such a way that each guy is sitting next to *at least* one gal, and *vice versa*. How many ways are there to do this? HINT: Use a decision tree, and practice by solving the 3-guy, 3-gal problem.

SOLUTION

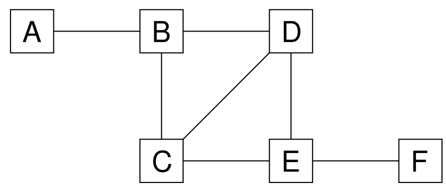
The tree below shows that there are 18 valid gender arrangements. It was developed by building stages describing the choices for each seat, subject to the constraint that each person is seated next to at least one person of the other gender (a "local" criterion). I then eliminated the paths that contain too many `B`*s or* `G`*s, leaving the nine valid paths. Someone else might have checked both conditions at once in developing their tree.*

Once the gender assignment has been made, there are $4!$ *ways to arrange the guys and* $4!$ *ways to arrange the gals; so the number of ways to assign the seats to individuals is*

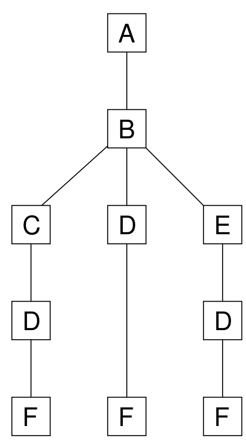
$$18 \cdot 4! \cdot 4!$$



2. Your neighborhood has a street map isomorphic to the undirected graph shown below. Use a decision tree count the number of ways to get from point A to point F without visiting the same point twice.



SOLUTION



3. A license plate consists of two letters of the alphabet followed by three decimal digits. How many different license plates are possible?

Solution

$26 \cdot 26 \cdot 10 \cdot 10 \cdot 10$

4. A committee of 12 is to be selected from 10 men and 10 women. In how many ways can the selection be carried out if

(a) There are no restrictions?

(b) There must be six men and six women?

(c) There must be an even number of women?

(d) There must be more women than men?

(e) There must be at least eight men?

Solution

(a) *If there are no restrictions:* $\binom{20}{12}$

(b) *If there must be six men and six women:* $\binom{10}{6} \cdot \binom{10}{6}$

(c) *If there must be an even number of women:*

$$\binom{10}{2} \cdot \binom{10}{10} + \binom{10}{4} \cdot \binom{10}{8} + \binom{10}{6} \cdot \binom{10}{6} + \binom{10}{8} \cdot \binom{10}{4} + \binom{10}{10} \cdot \binom{10}{2}$$

which simplifies to

$$2 \cdot \binom{10}{2} + 2 \cdot \binom{10}{4} \cdot \binom{10}{8} + \binom{10}{6} \cdot \binom{10}{6}$$

(d) *There must be more women than men:*

$$\binom{10}{7} \cdot \binom{10}{5} + \binom{10}{8} \cdot \binom{10}{4} + \binom{10}{9} \cdot \binom{10}{3} + \binom{10}{10} \cdot \binom{10}{2}$$

(e) *There must be at least eight men:*

$$\binom{10}{4} \cdot \binom{10}{8} + \binom{10}{3} \cdot \binom{10}{9} + \binom{10}{2} \cdot \binom{10}{10}$$

5. A hand of seven cards is dealt from a standard deck of 52 cards.

(a) In how many ways can the resulting hand contain two spades ($\spadesuit$) and five red cards ($\heartsuit$ or $\diamondsuit$)?

(b) The *likelihood* (probability) of an outcome with property P is the quotient of the number of outcomes with property P and the total number of outcomes. What is the likelihood that the hand described above will occur?

Solution

(a) *There are 13 spades and 26 red cards, and order doesn't matter, so the number of distinct hands is*

$$\binom{13}{2} \cdot \binom{26}{3}$$

(b) *There are* $\binom{52}{7}$ *different hands of seven cards. The liklihood, then is*

$$\frac{\binom{13}{2} \cdot \binom{26}{3}}{\binom{52}{7}}$$

Let's see if we can simplify (not necessary).

$$\frac{\dbinom{13}{2} \cdot \dbinom{26}{3}}{\dbinom{52}{7}}$$

$$= \frac{\dfrac{13!}{2! \cdot 11!} \cdot \dfrac{26!}{3! \cdot 23!}}{\dfrac{52!}{7! \cdot 45!}}$$

$$= \frac{\dfrac{13 \cdot 12}{2} \cdot \dfrac{26 \cdot 25 \cdot 24}{6}}{\dfrac{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48 \cdot 47 \cdot 46}{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2}}$$

$$= \frac{3 \cdot 5 \cdot 13}{47 \cdot 23 \cdot 17 \cdot 7}$$

6. A fair coin is tossed six times. How many ways can the six tosses result in two heads and four tails? How likely is this to happen?

Solution

The number of different outcomes is $2^6 = 64$. *The number of ways to assign heads to just two of the flips is* $\binom{6}{2} = 15$. *The likelihood of two heads and four tails, then, is*

$$\frac{15}{64}$$

Just for fun, here a listing of all the outcomes, with the successful ones indicated in red. You do not have to do this, or draw a counting tree, to have the correct answer.

1. hhhhhh
2. hhhhhT
3. hhhhTh
4. hhhhTT
5. hhhThh
6. hhhThT
7. hhhTTh
8. hhhTTT
9. hhThhh
10. hhThhT
11. hhThTh
12. hhThTT
13. hhTThh
14. hhTThT
15. hhTTTh
16. hhTTTT (1)
17. hThhhh
18. hThhhT
19. hThhTh
20. hThhTT
21. hThThh
22. hThThT
23. hThTTh
24. hThTTT (2)
25. hTThhh
26. hTThhT
27. hTThTh
28. hTThTT (3)
29. hTTThh
30. hTTThT (4)
31. hTTTTh (5)
32. hTTTTT
33. Thhhhh
34. ThhhhT
35. ThhhTh
36. ThhhTT
37. ThhThh
38. ThhThT
39. ThhTTh
40. ThhTTT (6)
41. ThThhh
42. ThThhT
43. ThThTh
44. ThThTT (7)
45. ThTThh
46. ThTThT (8)
47. ThTTTh (9)
48. ThTTTT
49. TThhhh
50. TThhhT
51. TThhTh
52. TThhTT (10)
53. TThThh
54. TThThT (11)
55. TThTTh (12)
56. TThTTT
57. TTThhh
58. TTThhT (13)
59. TTThTh (14)
60. TTThTT
61. TTTThh (15)
62. TTTThT
63. TTTTTh
64. TTTTTT

7.

(a) How many arrangements are there for all the letters in SOCIOLOGICAL?

(b) In how many of those arrangements are A and G adjacent?

(c) In how many of the arrangements are all the vowels adjacent?

Solution

(a) *There are* $12!$ *permutations of the 12 letters in* SOCIOLOGICAL. *However, some of these permutations are identical because of the repeated letters, 3* O*s, 2* I*s, 2* C*s, and 2* L*s. Consider, for example, the word* SOCIOLOGICAL *represents* $3! \cdot 2! \cdot 2! \cdot 2!$ *permutations, depending on how the* O*s,* I*s,* C*s, and* L*s are arranged. So the total number of distinct arrangements is*

$$\frac{12!}{3! \cdot 2! \cdot 2! \cdot 2!} = \frac{12!}{48}$$

(b)
1. *There are* 2 *ways to arrange the* A *and the* G.
2. *Since the* A *and the* G *must be adjacent, it is as though we are permuting 11 items.*
3. *Divide out the redundant arrangements as before.*

The number of distinct arrangements, then, is

$$\frac{2 \cdot 11!}{3! \cdot 2! \cdot 2! \cdot 2!}$$

(c)
1. *Similar to (b), if the vowels are all together, it is as though we are permuting 7 items.*
2. *Then permute the six vowels.*
3. *Divide out the redundant arrangements of 3* O*s,* I*s,* C*s, and* L*s as before.*

The number of distinct arrangements, then, is

$$\frac{7! \cdot 6!}{3! \cdot 2! \cdot 2! \cdot 2!}$$

8. A classroom contains n students. Disregarding leap years, how likely is it that at least two of the student have the same birthday—that is, born on the same day of the month but possibly in different years?

HINTS:

(a) The likelihood (probability) of an outcome with property P is the quotient of the number of outcomes with property P and the total number of outcomes. In this problem the total number of outcomes is 365^n because each student's birthday could be any day of the year.

(b) If the likelihood of an outcome with property P is p then the likelihood of an outcome that *does not* have property P is $1 - p$.

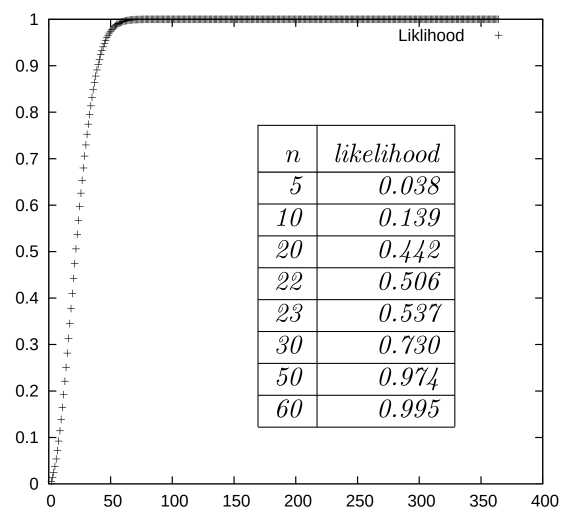
(c) Evaluating the liklihood for this problem is a messy job. If you want to give a numerical solution, rather than a formula, write a program to compute it. The numbers involved may be quite large, so beware of truncation errors. Remember that Scheme has "*bignums,*" and can do arithmetic on unbounded integers.

SOLUTION

If $n = 1$ there is a zero likelihood that two (different) students have the same birthday. If $n > 365$ it is certain that at least two students have the same birthday. So let us assume that $1 < n \leq 365$. It is easy to formulate an answer to the question, "How likely is it that no two of the n students has the same birthday*?" This is just an n-permutation on 365 dates. Hence, the resulting likelihood is*

$$\frac{\frac{365!}{n!}}{365^n} = \frac{365 \cdot 364 \cdot 363 \ \cdots \ (365 - n + 1)}{365^n}$$

The magnitudes of the numbers can make this a difficult number to compute. The graph to the right was generated by a Scheme program using 64-bit floating point approximations:



n	*likelihood*
5	*0.038*
10	*0.139*
20	*0.442*
22	*0.506*
23	*0.537*
30	*0.730*
50	*0.974*
60	*0.995*

Supplemental Problem. (Monty Hall Problem) This problem appeared in the *Ask Marylyn* (Marylyn Vos Savant) column of *Parade* Magazine on September 9, 1990. It has come to be called the *Monty Hall Problem.*[1]

You are a contestant in a game show called *Goat or No Goat*. In the final round, you are to select one of three doors to win the prize behind it. Behind two of the doors are goats, and behind the third is *A Brand New Car!*. The game works like this:

1. You choose a door.
2. The game-show host shows you which of two unchoosen door has a goat behind it.
3. You now have the option of sticking with your original choice or changing your choice to third, unopened door.

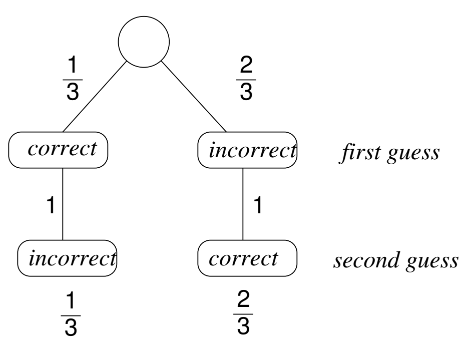
What should you do?

Solution

In her column, Vos Savant explained that you should always change your choice of doors at Step 3. This solution generated much controversy, due to misinterpretation the problem assumptions, namely that the host always shows you a door with a goat and *you decide beforehand what you are going to do—as you should.*

At Step 1 you have a 1/3 *chance of choosing the right door. If your strategy is to stick with that choice, your chance of winning is* 1/3. *Now suppose instead that your strategy is to change your choice. As the probability tree below shows,*

1. *If your first choice was correct (probability 1/3) you are certain to loose.*
2. *If your first choice was wrong (probability 2/3) you are certain to win because the host has shown you which of the two remaining doors has the car behind it.*



---

[1]Monty Hall was the original host of the game show *The Price Is Right*, after which *Goat or No Goat* is modelled