# C241 Homework Assignment 5

**1.** Label each step of the following logical derivations with its justification. A justification may one of the BOOLEAN IDENTITIES in Section 3.3, or a derived identitity from the textbook examples. Highlight the subexpression to which the identity is applied.

(a) $\overline{(p \cdot (q + r))}$

$= (\overline{p} + \overline{(q + r)}\ )$ — *DeMorgan*

$= \overline{p} + (\overline{q} \cdot \overline{r})$

(b) $\overline{((p \cdot q) \Rightarrow s)}$

$= \overline{(\overline{(p \cdot q)} + s)}$

$= \overline{\overline{(p \cdot q)}} \cdot \overline{s}$

$= (p \cdot q) \cdot \overline{s}$

$= (p \cdot q \cdot \overline{s})$

(c) $\overline{(p + q + (\overline{p} \cdot \overline{q} \cdot r))}$

$= \overline{((p + q) + (\overline{p} \cdot \overline{q} \cdot r))}$

$= (\overline{(p + q)} \cdot (\overline{\overline{p} \cdot \overline{q} \cdot r}))$

$= (\overline{(p + q)} \cdot (\overline{\overline{p}} + \overline{\overline{q}} + \overline{r}))$

$= (\overline{(p + q)} \cdot (p + q + \overline{r}))$

$= (\overline{(p + q)} \cdot ((p + q) + \overline{r}))$

$= (\overline{(p + q)} \cdot (p + q)) + (\overline{(p + q)} \cdot \overline{r})$

$= 0 + \ (\overline{(p + q)} \cdot \overline{r})$

$= (\overline{p + q}) \cdot \overline{r}$

$= (\overline{p} \cdot \overline{q}) \cdot \overline{r}$

$= \overline{p} \cdot \overline{q} \cdot \overline{r}$

(a)

| | |
|---|---|
| $\overline{(p \cdot (q + r))}$ | |
| $= (\overline{p} + \overline{(q + r)})$ | *DeMorgan* |
| $= \overline{p} + (\overline{q} \cdot \overline{r})$ | *DeMorgan* |

(b)

| | |
|---|---|
| $\overline{((p \cdot q) \Rightarrow s)}$ | |
| $= \overline{(\overline{(p \cdot q)} + s)}$ | $\neg(q + p)$ eq $(q \Rightarrow p)$ |
| $= \overline{\overline{(p \cdot q)}} \cdot \overline{s}$ | *DeMorgan* |
| $= (p \cdot q) \cdot \overline{s}$ | *negation* |
| $= (p \cdot q \cdot \overline{s})$ | *associativity* |

(c)

| | |
|---|---|
| $\overline{(p + q + (\overline{p} \cdot \overline{q} \cdot r))}$ | |
| $= \overline{((p + q) + (\overline{p} \cdot \overline{q} \cdot r))}$ | *associativity* |
| $= (\overline{(p + q)} \cdot (\overline{\overline{p} \cdot \overline{q} \cdot r}))$ | *DeMorgan* |
| $= (\overline{(p + q)} \cdot (\overline{\overline{p}} + \overline{\overline{q}} + \overline{r}))$ | *DeMorgan* |
| $= (\overline{(p + q)} \cdot (p + q + \overline{r}))$ | *negation, twice* |
| $= (\overline{(p + q)} \cdot ((p + q) + \overline{r}))$ | *associativity* |
| $= (\overline{(p + q)} \cdot (p + q)) + (\overline{(p + q)} \cdot \overline{r})$ | *distributivity* |
| $= 0 + (\overline{(p + q)} \cdot \overline{r})$ | *cancellation* |
| $= (\overline{p + q}) \cdot \overline{r}$ | *identity* |
| $= (\overline{p} \cdot \overline{q}) \cdot \overline{r}$ | *DeMorgan* |
| $= \overline{p} \cdot \overline{q} \cdot \overline{r}$ | *associativity* |

**2.** Use **only** the BOOLEAN IDENTITIES to prove that DeMogan's Laws are valid for three variables:

(a) $\overline{(p+q+r)} = (\overline{p} \cdot \overline{q} \cdot \overline{r})$

(b) $\overline{(p \cdot q \cdot r)} = (\overline{p} + \overline{q} + \overline{r})$

SOLUTION

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(a) $\overline{(p+q+r)}$

$= \overline{(p+(q+r))}$ *(associativity)*

$= (\overline{p} \cdot \overline{(q+r)})$ *(DeMorgan)*

$= (\overline{p} \cdot (\overline{q} \cdot \overline{r}))$ *(DeMorgan)*

$= (\overline{p} \cdot \overline{q} \cdot \overline{r})$ *(associativity)*

(b) *We know that every identity has a dual, so there is no need to perform the dual derivation. Here it is anyway.*

$= \overline{(p \cdot q \cdot r)}$

$= \overline{(p \cdot (q \cdot r))}$ *(associativity)*

$= (\overline{p} + \overline{(q \cdot r)})$ *(DeMorgan)*

$= (\overline{p} + (\overline{q} + \overline{r}))$ *(DeMorgan)*

$= (\overline{p} + \overline{q} + \overline{r})$ *(associativity)*

**3.** Define $x \oplus y$ to be $x\overline{y} + \overline{x}y$. Use boolean algebra to prove

(a) $x \oplus y = \overline{x} \oplus \overline{y}$

(b) $x(y \oplus z) = xy \oplus xz$

(c) $\overline{(x \oplus y)} = \overline{x} \oplus y$

SOLUTION

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(a) *Starting from the right-hand side,*

$$
\begin{aligned}
& \overline{x} \oplus \overline{y} \\
= \; & \overline{\overline{x}}\,\overline{y} + \overline{x}\,\overline{\overline{y}} && \textit{Defn. `}\oplus\textit{'} \\
= \; & x\overline{y} + \overline{x}\,y && \textit{Negation, twice} \\
= \; & \overline{x}\,y + x\,\overline{y} && \textit{Commutativity} \\
= \; & x \oplus y && \textit{Defn. `}\oplus\textit{'}
\end{aligned}
$$

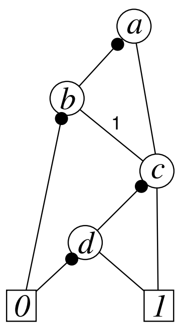
(b)

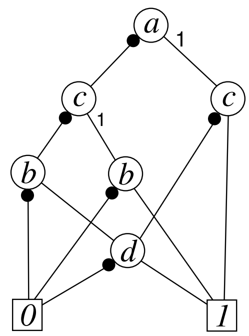
$$
\begin{aligned}
& x(y \oplus z) \\
= \; & x\,(\overline{y}\,z + y\,\overline{z}) && \textit{Defn. `}\oplus\textit{'} \\
= \; & x\,\overline{y}\,z + x\,y\,\overline{z} && \textit{Distributivity} \\
= \; & \overline{y}\,x\,z + x\,y\,\overline{z} && \textit{Rearrangement} \\
= \; & 0\,z + \overline{y}\,x\,z + 0\,y + x\,y\,\overline{z} && \textit{Dominance, Identity} \\
= \; & \overline{x}\,x\,z + \overline{y}\,x\,z + x\,y\,\overline{x} + x\,y\,\overline{z} && \textit{Cancellation} \\
= \; & (\overline{x} + \overline{y})\,x\,z + x\,y\,(\overline{x} + \overline{z}) && \textit{Distribuitivity} \\
= \; & \overline{x\,y}\,x\,z + x\,y\,\overline{x\,z} && \textit{Demorgan's Law} \\
= \; & x\,y \oplus x\,z && \textit{Defn. `}\oplus\textit{'}
\end{aligned}
$$

(c)

$$
\begin{aligned}
& \overline{(x \oplus y)} \\
= \; & \overline{\overline{x}\,y + x\,\overline{y}} && \textit{Defn. `}\oplus\textit{'} \\
= \; & \overline{\overline{x}\,y}\;\overline{x\,\overline{y}} && \textit{DeMorgan's Law} \\
= \; & (\overline{\overline{x}} + \overline{y})\,(\overline{x} + \overline{\overline{y}}) && \textit{DeMorgan's Law} \\
= \; & \overline{\overline{x}}\,\overline{x} + \overline{\overline{x}}\,\overline{\overline{y}} + \overline{y}\,\overline{x} + \overline{y}\,\overline{\overline{y}} && \textit{Distributivity} \\
= \; & x\,\overline{x} + \overline{\overline{x}}\,y + \overline{y}\,\overline{x} + \overline{Y}\,y && \textit{Negation} \\
= \; & 0 + \overline{\overline{x}}\,y + \overline{y}\,\overline{x} + 0 && \textit{Cancellation, twice} \\
= \; & \overline{\overline{x}}\,y + \overline{y}\,\overline{x} && \textit{Identity, twice} \\
= \; & \overline{\overline{x}}\,y + \overline{x}\,\overline{y} && \textit{Commutativity} \\
= \; & \overline{x} \oplus y && \textit{Defn. `}\oplus\textit{'}
\end{aligned}
$$

**4.** The ROBDDs below represent two terms over variables $a$, $b$, $c$ and $d$. In the one on the left, the variables are ordered $[a, b, c, d]$; and on the right the ordering is $[a, c, b, d]$. Determine whether or not the terms represented by these two graphs are equivalent.




SOLUTION

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*One way is to develop truth tables by following the ROBDD for each case. Alternatively, sum the terms for every path to* $\boxed{1}$*. This is similar to comparing DNFs. On the left we have*

$$\bar{a}\, b\, \bar{c}\, d + \bar{a}\, b\, c + a\, \bar{c}\, d + a\, c$$

*On the right,*

$$\begin{aligned} &\bar{a}\, \bar{c}\, b\, d + \bar{a}\, c\, b + a\, \bar{c}\, d + a\, c \\ &= \bar{a}\, b\, \bar{c}\, d + \bar{a}\, b\, c + a\, \bar{c}\, d + a\, c \end{aligned}$$

*So the two ROBDDs represent the same equivalence class of formulas under differing variable orderings.*

**5.** Use boolean algebra to synthesize the DNF for $(a+b)(c+\overline{a\,d})+\overline{bc}d$.

SOLUTION

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$$
\begin{aligned}
& (a+b)\,(c+\overline{a\,d})+\overline{b\,c}\,d \\
&= (a+b)\,(c+\bar{a}+\bar{d})+(\bar{b}+\bar{c})\,d && \textit{(DeMorgan } \times 2\textit{)} \\
&= (a+b)\,c+(a+b)\,\bar{a}+(a+b)\,\bar{d}+\bar{b}\,d+\bar{c}\,d && \textit{(distributivity } \times 2\textit{)} \\
&= a\,c+b\,c+a\,\bar{a}+b\,\bar{a}+a\,\bar{d}+b\,\bar{d}+\bar{b}\,d+\bar{c}\,d && \textit{(distributivity } \times 2\textit{)} \\
&= a\,c+b\,c+\bar{a}\,b+a\,\bar{d}+b\,\bar{d}+\bar{b}\,d+\bar{c}\,d && \textit{(cancellation, identity)}
\end{aligned}
$$

$$\boxed{a\,c \to a\,b\,c\,d+a\,b\,c\,\bar{d}+a\,\bar{b}\,c\,d+a\,b\,\bar{c}\,\bar{d}}$$

$$\boxed{b\,c \to \cancel{a\,b\,c\,d}+\cancel{a\,b\,c\,\bar{d}}+\bar{a}\,b\,c\,d+\bar{a}\,b\,c\,\bar{d}}$$

$$\boxed{\bar{a}\,b \to \cancel{\bar{a}\,b\,c\,d}+\bar{a}\,b\,\bar{c}\,d+\cancel{\bar{a}\,b\,c\,\bar{d}}+\bar{a}\,b\,\bar{c}\,\bar{d}}$$

$$\boxed{a\,\bar{d} \to \cancel{a\,b\,c\,\bar{d}}+\cancel{a\,b\,\bar{c}\,\bar{d}}+a\,\bar{b}\,c\,\bar{d}+\cancel{a\,\bar{b}\,\bar{c}\,\bar{d}}} \qquad \textit{(}x \to xy+x\bar{y}\textit{)}$$

$$\boxed{b\,\bar{d} \to \cancel{a\,b\,c\,\bar{d}}+\cancel{a\,b\,\bar{c}\,\bar{d}}+\cancel{\bar{a}\,b\,c\,\bar{d}}+\cancel{\bar{a}\,b\,\bar{c}\,\bar{d}}}$$

$$\boxed{\bar{b}\,d \to \cancel{a\,\bar{b}\,c\,d}+a\,\bar{b}\,\bar{c}\,d+\bar{a}\,\bar{b}\,c\,d+\bar{a}\,\bar{b}\,\bar{c}\,d}$$

$$\boxed{\bar{c}\,d \to a\,b\,\bar{c}\,d+\cancel{a\,\bar{b}\,\bar{c}\,d}+\cancel{\bar{a}\,b\,\bar{c}\,d}+\cancel{\bar{a}\,\bar{b}\,\bar{c}\,d}}$$

$$
\begin{aligned}
&= a\,b\,c\,d+a\,b\,c\,\bar{d}+a\,b\,\bar{c}\,d+a\,b\,\bar{c}\,\bar{d}+a\,\bar{b}\,c\,d+a\,\bar{b}\,c\,\bar{d} \\
&\quad +a\,\bar{b}\,\bar{c}\,d+a\,\bar{b}\,\bar{c}\,\bar{d}+\bar{a}\,b\,c\,d+\bar{a}\,b\,c\,\bar{d}+\bar{a}\,b\,\bar{c}\,d+\bar{a}\,b\,\bar{c}\,\bar{d} \\
&\quad +\bar{a}\,\bar{b}\,c\,d+\bar{a}\,\bar{b}\,\bar{c}\,d && \textit{(rearranging)}
\end{aligned}
$$

COMMENT. *If you didn't grind this out correctly (or if I didn't) there is no cause for concern. This is not the kind of thing humans do.*

**6.** A standard deck of cards has 52 cards consisting of 13 cards in each of four *suits*: $\spadesuit$, $\heartsuit$, $\diamondsuit$ $\clubsuit$. In each suit, cards have *face values* from $\{1, 2, \ldots, 13\}$, each card having a different face value. A *hand* is a set of five cards from the deck. A hand is called a *flush* if all five cards are of the same suit. A hand is called a *straight* if the five cards are sequential in value, for instance, $\{3\heartsuit, 4\spadesuit, 5\diamondsuit, 6\diamondsuit, 7\heartsuit\}$.

(a) How many different flushes are there in a standard deck?

(b) How many different straights are there in a standard deck?

Solution

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(a) *There are 4 suits to choose from. Having chosen a suit there are* $\binom{13}{5}$ *distinct subsets of 5 cards. Hence, the number of flushes is:*

$$4 \cdot \binom{13}{5} = \frac{4 \cdot 13!}{5! \cdot 8!} = 5,148$$

(b) *The possible sequences are* $\langle 1, 2, 3, 4, 5 \rangle$ *through* $\langle 9, 10, 11, 12, 13 \rangle$*, numbering 9 in all. For a given sequence, each card may be one of 4 suits. Hence the number of straights is* $9 \cdot 4^5 = 9,216$.

(c) *While we're at it, the number of* straight-flushes *is* $9 \cdot 4 = 36$.

**7.** Definition 4.1 (p. 63) states that for finite sets $A$ and $B$

$$|A \cup B| = |A| + |B| - |A \cap B|$$

(a) Write a corresponding formula for the union of three finite sets.

(b) Write a corresponding formula for the union of four finite sets.

(c) Write a formula for the union of $n$ finite sets.

SOLUTION

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(a)

$$\begin{aligned}
&|(A \cup B) \cup C| \\
&= |A \cup B| + |C| - |(A \cup B) \cap C| \\
&= (|A| + |B| - |A \cap B|) + |C| - |(A \cup B) \cap C| \\
&= |A| + |B| + |C| - |A \cap B| - |(A \cup B) \cap C| \\
&= |A| + |B| + |C| - |A \cap B| - |(A \cap C) \cup (B \cap C)| \\
&= |A| + |B| + |C| - |A \cap B| - (|A \cap C| + |B \cap C| - |(A \cap C) \cap (B \cap C)|) \\
&= |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |(A \cap C) \cap (B \cap C)| \\
&\stackrel{\star}{=} |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |(A \cap B \cap C)|
\end{aligned}$$

(b) *Using the result from (a)*

$$\begin{aligned}
&|A \cup B \cup (C \cup D)| \\
&\stackrel{\star}{=} |A| + |B| + |C \cup D| - |A \cap B| - |A \cap (C \cup D)| - |B \cap (C \cup D)| + |A \cap B \cap (C \cup D)|
\end{aligned}$$

$$\begin{aligned}
+\,|C \cup D| &= |C| + |D| - |C \cap D| \\
-\,|A \cap (C \cup D)| &= -|(A \cap C) \cup (A \cap D)| \\
&= -|A \cap C| - |A \cap D| + |A \cap C \cap D| \\
-\,|B \cap (C \cup D)| &= -|(B \cap C) \cup (B \cap D)| \\
&= -|B \cap C| - |B \cap D| + |B \cap C \cap D| \\
+\,|A \cap B \cap (C \cup D)| &= |(A \cap B \cap C) \cup (A \cap B \cap D)| \\
&= |A \cap B \cap C| + |A \cap B \cap D| - |A \cap B \cap C \cap D|
\end{aligned}$$

*Putting all this together,*

$$\begin{aligned}
|A \cup B \cup C \cup D| = {} & |A| + |B| + |C| + |D| \\
& -|A \cap B| - |A \cap C| - |A \cap D| - |B \cap C| - |B \cap D| - |C \cap D| \\
& +|A \cap B \cap C| + |A \cap B \cap D| - |A \cap C \cap D| - |B \cap C \cap D| \\
& -|A \cap B \cap C \cap D|
\end{aligned}$$

(c)

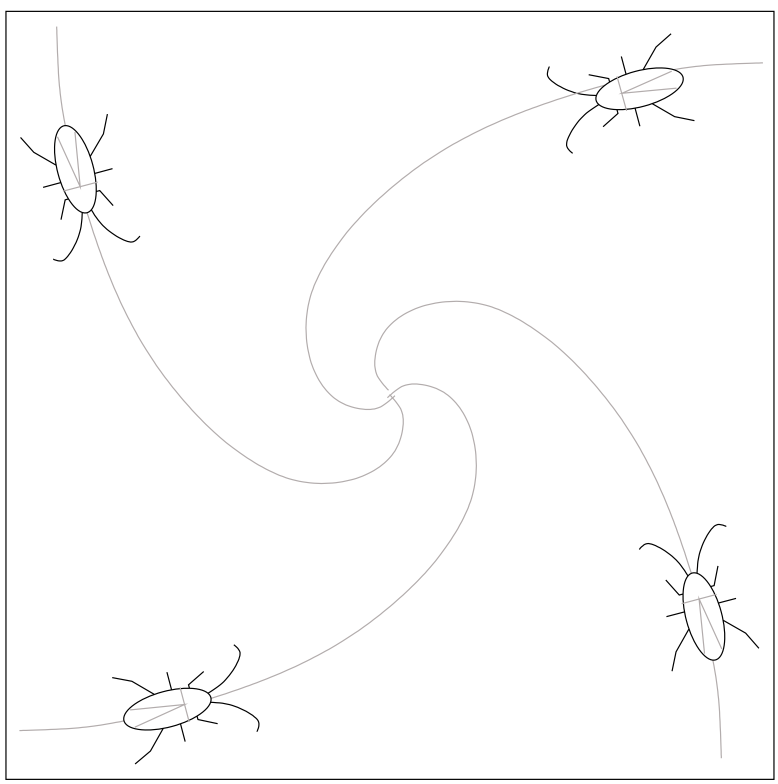
$$\left| \bigcup_{1 \le i \le n} A_i \right| = \sum_{1 \le i \le n} |A_i| - \sum_{1 \le i < j \le n} |A_i \cap A_j| + \sum_{1 \le i < j < k \le n} |A_i \cap A_j \cap A_k| - \ldots + (-1)^n |A_1 \cap \cdots \cap A_n|$$

Supplemental Problem. (Amorous Cockroaches) Comment: *This is neither a calculus question nor a logic puzzle.*

At last! You've gotten out of the dorms and into an apartment. Life can begin. You get moved in and a few days later are looking for a brownie pan. When you open the cabinet door, you see a 12-inch by 12-inch pan, just perfect for your cooking needs. That's the good news.

Unfortunately, there are four cockroaches in the pan, one at each corner. Now you may not have known this, but cockroaches are very amorous creatures, and will always move directly toward the object of their affection. Each of the four in your pan is attracted to the one in the adjacent corner, going counter-clockwise. So all four cockroaches simultaneously start walking toward the one that attracts them. As a result, they start spiraling toward the center of the pan (See the diagram below).

Question: *How far will the cockroaches travel?*



## Solution

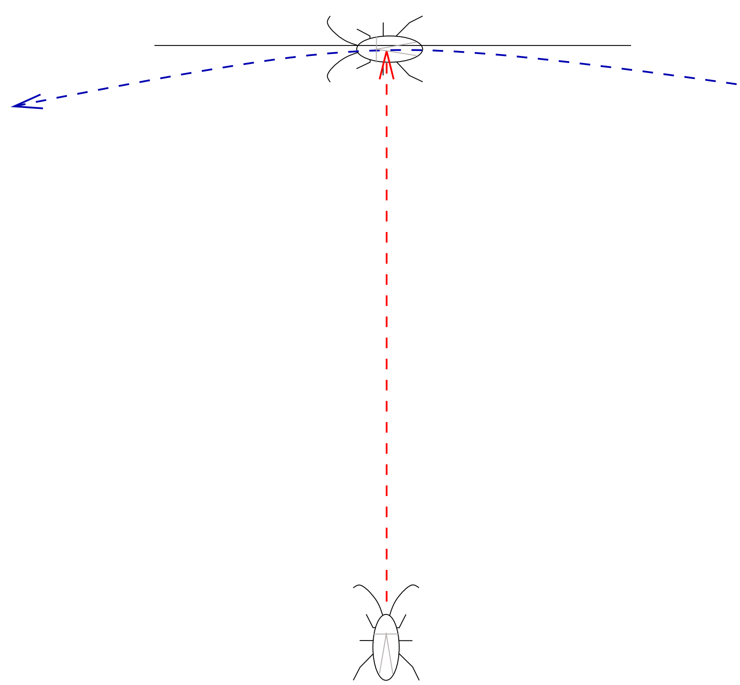
*Each cockroach travels the length of one side of the square pan, 12 inches in the problem as stated.*

*To gain the necessary insight, you have to look at the problem from the (any) cockroach's point of view. From that perspective,*

(a) *the cockroach is moving directly toward its target, perpendicular to the (tangent) of the target's path of travel. We know from geometry that this is the shortest path it can take.*

(b) *We also know (from common sense), that the* total distance *a cockroach travels is* $d = \alpha + \beta$*, where* $\alpha$ *is the distance it has already traveled and* $\beta$ *is how far it has yet to go. When the cockroach takes a step,* $\alpha$ *and* $\beta$ *change by the same distance* $s$*, but* $d = (\alpha + s) + (\beta - s)$ *is is unchanged.*

*These two properties are* invariant*—they do not change throughout cockroach's trip from the corner of the pan to its destination. From the cockroach's point of view, it is the pan that is turning, not itself. So by (a), above, it is invariantly traveling along the shortest path to its destination; and the* total distance *(b) is invariant also. Initially,* $d = 12$ *inches, so that is how far it travels.*



REMARK. *The* COMMENT *at the beginning of the problem says that you don't need calculus to solve the problem, but this is not entirely true. What you do need is a grasp of the underlying concepts of calculus. The solution is only valid in the limit. The true distance is some multiple of the number of discrete steps taken by the cockroach. Twelve inches is the limit of that number as the length of each stride becomes infinitesimal.*[1]

*One can reasonably argue (as I suspect your Calculus instructor would) that the "solution" given above needs to be* verifyed *by a rigorous mathematical formulation by which the limit value can be* calcul*ated. It is merely an idea about how to approach the* proof. *It is worthwhile to set up that formulation. One the other hand, if you were to write a program to approximate the limit, discovering the problem's* invariants *key to the* comput*ing the answer.*

[1]Here's an interesting fact about cockroaches. Most insects have two forms of locomotion or *gaits*. One is *walking* in which two outer legs and to opposite middle leg work in unison. The three legs act as a tripod, and the two tripods alternate, propelling the the insect forward in a bi-pedal manner. The other gait is *metachronal*: the legs on either side thrust in sequence, resulting in a "wave" traversing from front to back, like the legs of a centipede or caterpillar. Cockroaches are unusual insects in that they have *three* gaits. In flight, a cockroach can literally *run* on its two hind legs, keeping the other four raised off the ground.