C241 Homework Assignment 4

1. Which of the following formulas are tautologies and which are contradictions? Which of the formulas are logically equivalent to each other?

(a) $p \wedge (q \vee r)$

(b) $(\neg p \wedge r) \Rightarrow (q \vee r)$

(c) $(p \wedge q) \vee (p \vee r)$

(d) $\neg(r \Rightarrow q \wedge r)$

(e) $\neg(p \Rightarrow (q \Rightarrow p)$

(f) $((p \Rightarrow q) \vee (r \wedge s \vee t)) \vee (p \wedge \neg q)$

Solution

As demonstrated by their truth tables below, (a), (c) and (d) are contigencies; (b) and (f) are tautologies; (e) is a contradiction. None of the formulas are equivalent.

(a)

p	q	r	p	$\wedge$	$(q$	$\vee$	$r)$
F	F	F	F	F	F	F	F
F	F	T	F	F	F	T	T
F	T	F	F	F	T	T	F
F	T	T	F	F	T	T	T
T	F	F	T	F	F	F	F
T	F	T	T	T	F	T	T
T	T	F	T	T	T	T	F
T	T	T	T	T	T	T	T

(b)

p	q	r	$(\neg$	p	$\wedge$	$r)$	$\Rightarrow$	$(q$	$\vee$	$r)$
F	F	F	T	F	F	F	T	F	F	F
F	F	T	T	F	T	F	T	T	T	F
F	T	F	T	F	F	T	T	F	T	T
F	T	T	T	F	T	T	T	T	T	T
T	F	F	F	T	F	F	T	F	F	F
T	F	T	F	T	F	F	T	T	T	F
T	T	F	F	T	F	T	T	F	T	T
T	T	T	T	T	F	T	T	T	T	T

(c)

p	q	r	$((p$	$\wedge$	$q)$	$\vee$	$(p$	$\vee$	$r))$
T	F	F	F	F	F	F	F	F	F
T	F	T	F	F	F	T	F	T	T
T	T	F	F	F	T	F	F	F	F
T	T	T	F	F	T	T	F	T	T
T	F	F	T	F	F	T	T	T	F
T	F	T	T	F	F	T	T	T	T
T	T	F	T	T	T	T	T	T	F
T	T	T	T	T	T	T	T	T	T

(d)

q	r	$\neg$	$(r$	$\Rightarrow$	$(q$	$\wedge$	$r)))$
F	F	F	F	T	F	F	F
F	T	F	F	T	T	F	F
T	F	T	T	F	F	F	T
T	T	F	T	T	T	T	T

(e)

P	Q	$\neg$	$(p$	$\Rightarrow$	$(q$	$\Rightarrow$	$p)))$
F	F	F	F	T	F	T	F
F	F	F	F	T	T	F	F
T	F	F	T	T	F	T	T
T	T	F	T	T	T	T	T

(f) *The truth table is large enough that it pays to analyze the expression first. If you recognize that* $\neg(p \Rightarrow q)$ eq $(p \wedge \neg q)$ *then this formula is an instance of* $X \vee Y \vee \neg X$ *for* $X \equiv (p \Rightarrow q)$ *and* $Y \equiv ((r \wedge s) \vee t)$. *Since*

$$X \vee Y \vee \neg X \quad \text{eq} \quad (X \vee \neg X) \vee Y \quad \text{eq} \quad T \vee Y \quad \text{eq} \quad T$$

the whole formula is a tautology.

p	q	r	s	t	$(((p$	$\Rightarrow$	$q)$	$\vee$	$((r$	$\wedge$	$s)$	$\vee$	$t))$	$\vee$	$(p$	$\wedge$	$\neg$	$q))$
F	F	F	F	F	F	T	F	T	F	F	F	F	F	T	F	F	T	F
F	F	F	F	T	F	T	F	T	F	F	F	T	T	T	F	F	T	F
F	F	F	T	F	F	T	F	T	F	F	T	F	F	T	F	F	T	F
F	F	F	T	T	F	T	F	T	F	F	T	T	T	T	F	F	T	F
F	F	T	F	F	F	T	F	T	T	F	F	F	F	T	F	F	T	F
F	F	T	F	T	F	T	F	T	T	F	F	T	T	T	F	F	T	F
F	F	T	T	F	F	T	F	T	T	T	T	T	F	T	F	F	T	F
F	F	T	T	T	F	T	F	T	T	T	T	T	T	T	F	F	T	F
F	T	F	F	F	F	T	T	T	F	F	F	F	F	T	F	F	F	T
F	T	F	F	T	F	T	T	T	F	F	F	T	T	T	F	F	F	T
F	T	F	T	F	F	T	T	T	F	F	T	F	F	T	F	F	F	T
F	T	F	T	T	F	T	T	T	F	F	T	T	T	T	F	F	F	T
F	T	T	F	F	F	T	T	T	T	F	F	F	F	T	F	F	F	T
F	T	T	F	T	F	T	T	T	T	F	F	T	T	T	F	F	F	T
F	T	T	T	F	F	T	T	T	T	T	T	T	F	T	F	F	F	T
F	T	T	T	T	F	T	T	T	T	T	T	T	T	T	F	F	F	T
T	F	F	F	F	T	F	F	F	F	F	F	F	F	T	T	T	T	F
T	F	F	F	T	T	F	F	T	F	F	F	T	T	T	T	T	T	F
T	F	F	T	F	T	F	F	F	F	F	T	F	F	T	T	T	T	F
T	F	F	T	T	T	F	F	T	F	F	T	T	T	T	T	T	T	F
T	F	T	F	F	T	F	F	F	T	F	F	F	F	T	T	T	T	F
T	F	T	F	T	T	F	F	T	T	F	F	T	T	T	T	T	T	F
T	F	T	T	F	T	F	F	T	T	T	T	T	F	T	T	T	T	F
T	F	T	T	T	T	F	F	T	T	T	T	T	T	T	T	T	T	F
T	T	F	F	F	T	T	T	T	F	F	F	F	F	T	T	F	F	T
T	T	F	F	T	T	T	T	T	F	F	F	T	T	T	T	F	F	T
T	T	F	T	F	T	T	T	T	F	F	T	F	F	T	T	F	F	T
T	T	F	T	T	T	T	T	T	F	F	T	T	T	T	T	F	F	T
T	T	T	F	F	T	T	T	T	T	F	F	F	F	T	T	F	F	T
T	T	T	F	T	T	T	T	T	T	F	F	T	T	T	T	F	F	T
T	T	T	T	F	T	T	T	T	T	T	T	T	F	T	T	F	F	T
T	T	T	T	T	T	T	T	T	T	T	T	T	T	T	T	F	F	T

2. Use a truth table to show that

(a) $\neg(p \vee q)$ is <u>*not*</u> logically equivalent to $(\neg p \vee \neg q)$, and

(b) $\neg(p \wedge q)$ is <u>*not*</u> logically equivalent to $(\neg p \wedge \neg q)$.

Remember these facts.

SOLUTION

(a)

p	q	$\neg(p \vee q)$	$\neg p \vee \neg q$
F	F	T	T
F	T	F	T
T	F	F	T
T	T	F	F

(b)

p	q	$\neg(p \wedge q)$	$\neg p \wedge \neg q$
F	F	T	T
F	T	T	F
T	F	T	F
T	T	F	F

3. Consider the logical operation defined below:

P	Q	$P \downarrow Q$
F	F	T
F	T	T
T	F	T
T	T	F

Show that '$\downarrow$' can be used to implement (in the sense of Prop. 3.2) all of the operations of Definition 3.1.

Solution

By Proposition 3.2, all the operations can be implemented with '$\vee$' and '$\neg$', so it suffices to show how these can be implemended with '$\downarrow$'.

$$
\begin{array}{rcl}
\neg P & \rightarrow & P \downarrow P \\
P \vee Q & \rightarrow & (P \downarrow P) \downarrow (Q \downarrow Q) \\
\hline
P \wedge Q & \rightarrow & (P \downarrow Q) \downarrow (P \downarrow Q) \\
P \Rightarrow Q & \rightarrow & P \downarrow (Q \downarrow Q) \\
P \Leftrightarrow Q & \rightarrow & ((P \downarrow P) \downarrow (Q \downarrow Q)) \downarrow (P \downarrow Q) \\
P \not\Leftrightarrow Q & \rightarrow & (P \downarrow (Q \downarrow Q)) \downarrow (P \downarrow Q)
\end{array}
$$

4. Let P stand for the proposition "Sue says it." Let Q stand for the proposition "Sam saw it." Let R stand for the proposition "Sid did it." Express the following sentences as formulas involving the logical connectives. If there is more than one way to translate a sentence, use truth tables to explain any differences in the meaning among these translations.

(a) Sid did it, Sam saw it, and Sue says it.

(b) If Sid did it, Sam saw it.

(c) Sid did it only if Sam saw it.

(d) Sue says it only if Sid did it, and Sam saw it.

(e) If Sue says it implies Sam saw it, Sid did it.

SOLUTION

(a) $R \wedge Q \wedge P$

(b) $R \Rightarrow Q$

(c) $R \Rightarrow Q$

(d) *The comma indicates that the interpretation should be* $(P \Rightarrow R) \wedge Q$. *Ignoring the comma, another possibility is* $P \Rightarrow (R \wedge Q)$. *The truth table shows that these are distinct interpretations.*

P	Q	R	$(P \Rightarrow R) \wedge Q$	$P \Rightarrow (R \wedge Q)$
F	F	F	F	T
F	F	T	F	T
F	T	F	T	T
F	T	T	T	T
T	F	F	F	F
T	F	T	F	F
T	T	F	F	T
T	T	T	T	T

(e) $(P \Rightarrow Q) \Rightarrow R$, *which is not the same as* $P \Rightarrow (Q \Rightarrow R)$

P	Q	R	$(P \Rightarrow Q) \Rightarrow R$	$P \Rightarrow (Q \Rightarrow R)$
F	F	F	F	T
F	F	T	T	T
F	T	F	F	T
F	T	T	T	T
T	F	F	T	T
T	F	T	T	T
T	T	F	F	F
T	T	T	T	T

5. Determine whether the following proposition is a tautology.

$$(a \vee b \Leftrightarrow c) \wedge (d \vee e) \Leftrightarrow ((a \vee b \Leftrightarrow c) \wedge d) \vee ((a \vee b \Leftrightarrow c) \wedge e)$$

Solution

The problem is simpler to solve if you recognize that a single term occurs three times in the proposition:

$$\boxed{(a \vee b \Leftrightarrow c)} \wedge (d \vee e) \Leftrightarrow (\boxed{(a \vee b \Leftrightarrow c)} \wedge d) \vee (\boxed{(a \vee b \Leftrightarrow c)} \wedge e)$$

Call this repeated term X. It should suffice to determine whether

$$X \wedge (d \vee e) \Leftrightarrow (X \wedge d) \vee (X \wedge e)$$

is a tautology because the orginal proposiiton can be obtained by substituting term $\forall$ for the X. The truth table for this term has eight cases, where the original had thirty-two.

X	d	e	X	$\wedge$	$(d \vee e)$	$\Leftrightarrow$	$(X \wedge d)$	$\vee$	$(X \wedge e)$
F	F	F	F	F	F	T	F	F	F
F	F	T	F	F	T	T	F	F	F
F	T	F	F	F	T	T	F	F	F
F	T	T	F	F	T	T	F	F	F
T	F	F	T	F	F	T	F	F	F
T	F	T	T	T	T	T	F	T	T
T	T	F	T	T	T	T	T	T	F
T	T	T	T	T	T	T	T	T	T

Thus, the proposition is a tautology. By the way, we have just shown that '$\wedge$' distributes over '$\vee$', in the same way that multiplication distributes over addition.

In general we should check whether $a \vee b \Leftrightarrow c$ is a tautology or a contradiction. In either case the problem is easier because the truth table is smaller. But it doesn't matter what X evaluates to; $X \wedge (d \vee e) \Leftrightarrow (X \wedge d) \vee (X \wedge e)$ still evaluates to T.

6. Show whether the following pairs of formulas are equivalent.

(a) $(p \Rightarrow q) \Rightarrow r$ and $p \Rightarrow (q \Rightarrow r)$

(b) $p \Rightarrow (q \Rightarrow r)$ and $(p \wedge q) \Rightarrow r$

(c) $(p \wedge q) \Rightarrow r$ and $(p \Rightarrow r) \wedge (q \Rightarrow r)$

SOLUTION

(a) $(p \Rightarrow q) \Rightarrow r$ *and* $p \Rightarrow (q \Rightarrow r)$

Since their truth tables differ, these formulas are not equivalent

(a)

p	q	r	$(p \Rightarrow q)$	$\Rightarrow$	r	p	$\Rightarrow$	$(q \Rightarrow r)$	
T	T	T	T	T	T	T	T	T	
T	T	F	T	F	F	T	F	F	
T	F	T	F	T	T	T	T	T	
T	F	F	F	T	F	T	T	T	
F	T	T	T	T	T	F	T	T	
F	T	F	T	F	F	F	T	F	$\leftarrow$
F	F	T	T	T	T	F	T	T	
F	F	F	T	F	F	F	T	T	

(b) $p \Rightarrow (q \Rightarrow r)$ *and* $(p \wedge q) \Rightarrow r$

These formulas are equivalent; their truth tables are the same

p	q	r	p	$\Rightarrow$	$(q \Rightarrow r)$	$(p \wedge q)$	$\Rightarrow$	r
T	T	T	T	T	T	T	T	T
T	T	F	T	F	F	T	F	F
T	F	T	T	T	T	F	T	T
T	F	F	T	T	T	F	T	F
F	T	T	F	T	T	F	T	T
F	T	F	F	T	F	F	T	F
F	F	T	F	T	T	F	T	T
F	F	F	F	T	F	F	T	F

(c) $(p \wedge q) \Rightarrow r$ *and* $(p \Rightarrow r) \wedge (q \Rightarrow r)$

Since their truth tables differ, these formulas are not equivalent

p	q	r	$(p \wedge q)$	$\Rightarrow$	r	$(p \Rightarrow r)$	$\wedge$	$(q \Rightarrow r)$	
T	T	T	T	T	T	T	T	T	
T	T	F	T	F	F	F	F	F	
T	F	T	F	T	T	T	T	T	
T	F	F	F	T	F	F	F	T	$\leftarrow$
F	T	T	F	T	T	T	T	T	
F	T	F	F	T	F	T	F	F	$\leftarrow$
F	F	T	F	T	T	T	T	T	
F	F	F	F	T	F	T	T	T	

7. For each of the following propositions, give the DNF under the variable ordering $\langle a, b, c\rangle$.

(a) $a \vee (\neg a \wedge \neg b)$

(b) $a \Rightarrow (b \Leftrightarrow c)$

(c) $(\neg b \wedge c) \wedge (\neg a \Rightarrow \neg c) \wedge (c \wedge (\neg b \vee \neg a))$

(d) $(a \Rightarrow b) \Leftrightarrow (b \Rightarrow c)$

SOLUTION

(a) *Variable c does not appear in this formula, so it may be excluded from the truth table.*

a	b	c	a	$\vee$	$(\neg a$	$\wedge$	$\neg b)$	
F	F	–		T		T		←
F	T	–		F		F		
T	F	–		T		F		←
T	T	–		T		F		←

However, a DNF ovar $\langle a, b, c\rangle$ might be required to include all c-clauses. This "full" DNF is

$$(\neg a\wedge\neg b\wedge\neg c)\vee(\neg a\wedge\neg b\wedge c)\vee(\neg a\wedge b\wedge\neg c)\vee(\neg a\wedge b\wedge c)\vee(\neg a\wedge\neg b\wedge\neg c)\vee(a\wedge b\wedge\neg c)\vee(a\wedge b\wedge c)$$

If c is omitted, the resulting DNF is

$$(\neg a \wedge \neg b) \vee (\neg a \wedge b) \vee (a \wedge b)$$

(b)

a	b	c	$a \Rightarrow (b \Leftrightarrow c)$	
F	F	F	T	←
F	F	T	T	←
F	T	F	T	←
F	T	T	T	←
T	F	F	T	←
T	F	T	F	
T	T	F	F	
T	T	T	T	←

The DNF is

$$(\neg a\wedge\neg b\wedge\neg c)\vee(\neg a\wedge\neg b\wedge c)\vee(\neg a\wedge b\wedge\neg c)\vee(\neg a\wedge b\wedge c)\vee(a\wedge\neg b\wedge\neg c)\vee(a\wedge b\neg c)$$

(c) *In the truth table below, working from left to right, I stopped once I reached a conjunct that evaluates to F. The red entries show where I stopped.*

a	b	c	$(\neg b \wedge c)$	$\wedge$	$(\neg a \Rightarrow \neg c)$	$\wedge$	$(c \wedge (\neg b \vee \neg a))$
F	F	F	F	F	T	F	F
F	F	T	T	F	F	F	T
F	T	F	F	F	T	F	F
F	T	T	F	F	F	F	T
T	F	F	F	F	T	F	F
T	F	T	T	T	T	T	T
T	T	F	F	F	T	F	F
T	T	T	F	F	T	F	F

The DNF contains no clauses. The "or" of zero terms evaluates to F, so one could write $DNF \equiv T$.

(d)

a	b	c	$(a \Rightarrow b)$	$\Leftrightarrow$	$(b \Rightarrow c)$	
F	F	F	T	T	T	$\leftarrow$
F	F	T	T	T	T	$\leftarrow$
F	T	F	T	F	F	
F	T	T	T	T	T	$\leftarrow$
T	F	F	F	F	T	
T	F	T	F	F	T	
T	T	F	T	F	F	
T	T	T	T	T	T	$\leftarrow$

$$DNF \equiv (\neg a \wedge \neg b \wedge \neg c) \vee (\neg a \wedge \neg b \wedge c) \vee (\neg a \wedge b \wedge c) \vee (a \wedge b \wedge c)$$

8. Reduce the following boolean expressions to simpler terms

(a) $xy + (x+y)\overline{z} + y$

(b) $x + y + \overline{(\overline{x} + y + z)}$

(c) $yz + wx + z + [wz(xy + wz)]$

SOLUTION

(a) $xy + (x+y)\overline{z} + y$

$= xy + y + (x+y)\overline{z}$ *commutativity*

$= xy + 1y + (x+y)\overline{z}$ *Identity*

$= yx + y1 + (x+y)\overline{z}$ *commutativity, twice*

$= y(x+1) + y\overline{z} + x\overline{z}$ *distributivity*

$= y1 + y\overline{z} + x\overline{z}$ *dominance*

$= y(1+\overline{z}) + x\overline{z}$ *distributivity*

$= y1 + x\overline{z}$ *dominance*

$= y + x\overline{z}$ *identity*

For the remaining derivations, rearrangement by commutativity and associativity is implicit.

(b) $x + y + \overline{(\overline{x} + y + z)}$

$= x + y + (\overline{\overline{x}}\ \overline{y}\ \overline{z})$ *DeMorgan's Law*

$= x + y + (x\ \overline{y}\ \overline{z})$ *negation*

$= y + x + (x\ \overline{y}\ \overline{z})$ *commutativity*

$= y + x\ 1 + x\ \overline{y}\ \overline{z}$ *identity*

$= y + x\ (1 + \overline{y}\ \overline{z})$ *distributivity*

$= y + x$ *dominance, identity*

(c) $yz + wx + z + [wz(xy + wz)]$

$= yz + wx + z + wzxy + wzwz$ *distributivity, twice*

$= yz + wx + z + wzxy + wz$ *idempotence*

$= wx + z + yz + wzxy + wz$ *commutativity, associativity*

$= wx + z(1 + y + wxy + w)$ *distributivity*

$= wx + z$ *dominance, identity*

9. Write the truth tables for the following logical formulas and state whether each is a tautology, a contradiction, or neither (a contingency).

(a) $P \wedge (Q \vee R)$

(b) $(P \wedge \neg P) \Rightarrow Q$

(c) $P \Rightarrow (Q \vee \neg Q)$

SOLUTION

				(a)			(b)			(c)	
R	P	Q	P	$\wedge$	$(Q \vee R)$	$(P \wedge \neg P)$	$\Rightarrow$	Q	P	$\Rightarrow$	$(Q \vee \neg Q)$
T	T	T	T	T	T	F	T	T	T	T	T
T	T	F	T	T	T	F	T	F	T	T	T
T	F	T	T	T	T	F	T	T	F	T	T
T	F	F	T	F	F	F	T	F	F	T	T
F	T	T	F	F	T	–	–			–	–
F	T	F	F	F	F	–	–			–	–
F	F	T	F	F	T	–	–			–	–
F	F	F	F	F	F	–	–			–	–
				contigency			*tautology*			*tautology*	

COMMENTS.

1. *I chose to arrange the truth table variables in the order R, P, Q because the formula in Parts (b) and (c) contain only P and Q. This way, I could use just the upper half of the table as a truth table for (c).*

2. *In Part (b) once I recognized that the premise $P \wedge \neg P$, is a contradiction, I realized that evaluation of the implication $(P \wedge \neg P) \Rightarrow Q$ "reduces" to $F \Rightarrow Q$, which is always T.*

3. *Similarly, in Part (c), once I recognized that the conclusion $Q \vee \neg Q$ is always T, and $P \Rightarrow T$ is always T*

No SUPPLEMENTAL PROBLEM this week.