C241 Homework Assignment 3

1. Let

$$\begin{aligned} A &= \{a, b, c, d, e\} \\ R &= \{(a,b), (a,c), (b,a), (c,a), (c,d), (c,e), (d,c), (e,c)\} \end{aligned}$$

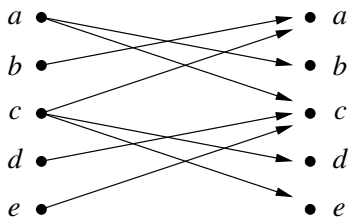
(a) Draw the bipartite graph representation of R.

(b) Draw the directed graph representation of R.

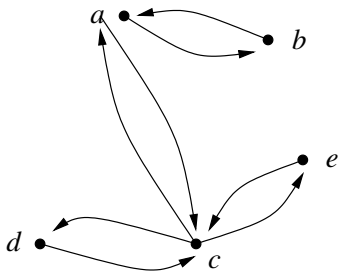
(c) Is R symmetric? reflexive? transitive?

SOLUTION

(a) *Draw the bipartite graph representation of R.*



(b) *Draw the directed graph representation of R.*



(c) *Is R symmetric? reflexive? transitive?*

1. *R is symmetric. For every edge (x, y) the edge (y, x) is also in R.*
2. *R is not reflexive. In fact it is irreflexive: there is <u>no</u> edge $(x, x) \in R$.*
3. *R is not transitive. One counterexample is: $(a, b) \in R$ and $(b, a) \in R$ but $(a, a) \notin R$.*

2. Let $A = \{a\}$, $B = \{a, b\}$.

(a) List all the relations $R \subseteq A \times A$.

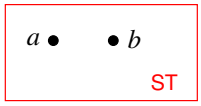
(b) List all the relations $R \subseteq B \times B$.

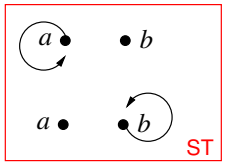
(c) Of the relations in (a) and (b), which are reflexive? Symmetric? Tranansitive?

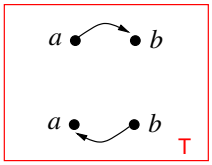
SOLUTION

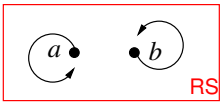
{a}x {a}

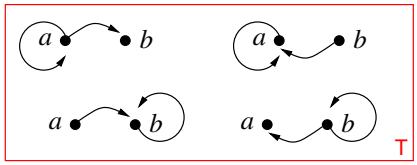
{a, b}x {a, b}

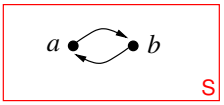


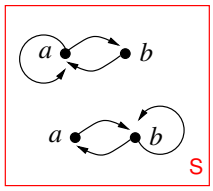


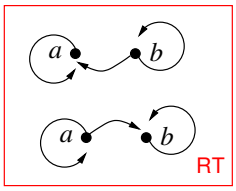


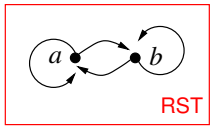






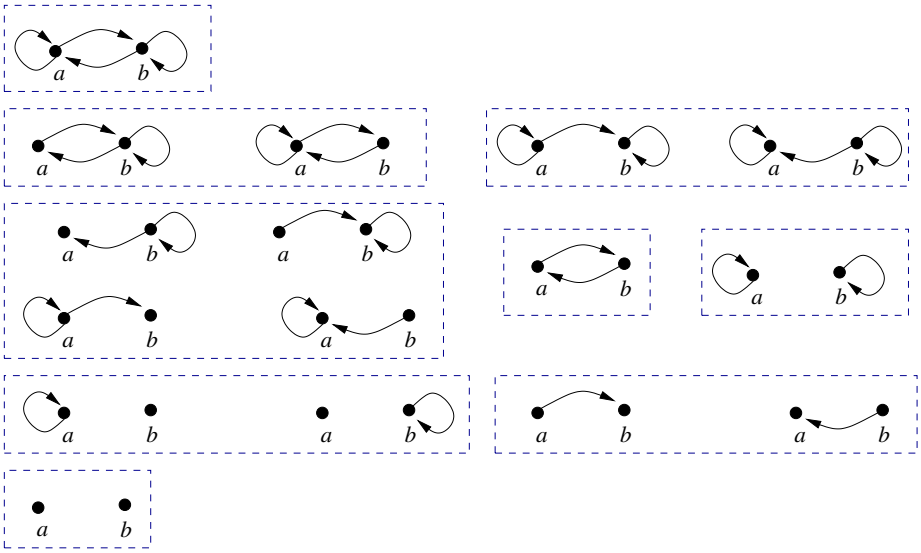




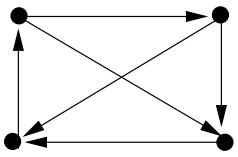


3. Draw all the directed graphs on a set with two elements. Indicate which of these graphs are isomorphic to one another.

SOLUTION



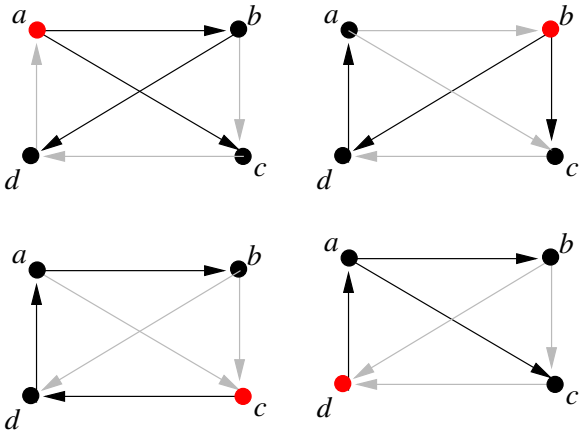
4. Recall (Definition 2.14) that $G \subseteq A \times A$ is a *rooted graph* iff there is a node $r \in A$ such that for every $x \in A$ there is a path from r to x in G. In the graph to the right, which nodes can serve as r in this definition?



SOLUTION

Any of the nodes a, b and d can serve as roots. Let $A = \{a, b, c, d\}$ as shown below.

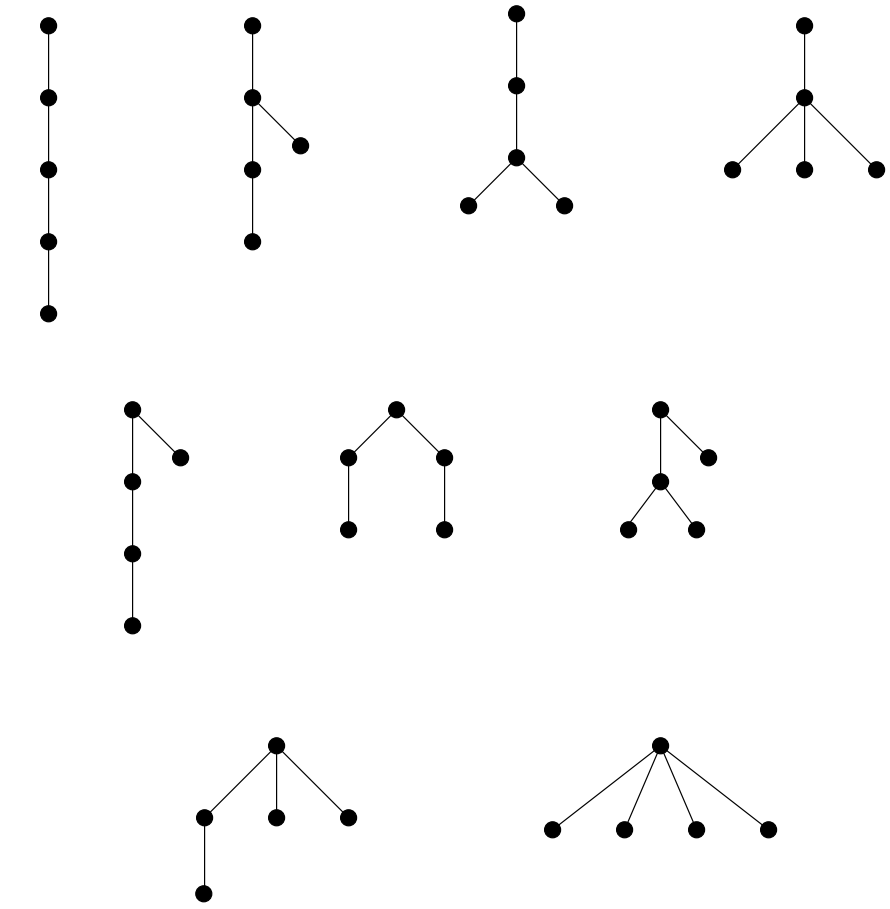
- *A spanning tree rooted at a is $\{(a, b), (a, c), (b, d)\}$.*
- *A spanning tree rooted at b is $\{(b, d), (b, c), (d, a)\}$.*
- *A spanning tree rooted at c is $\{(c, d), (d, a), (a, b)\}$.*
- *A spanning tree rooted at d is $\{(d, a), (a, b), (a, c)\}$.*



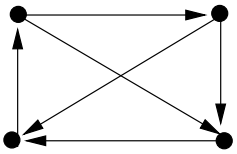
There are several more spanning trees possible.

5. Draw all the nonisomorphic trees with five vertices.

SOLUTION

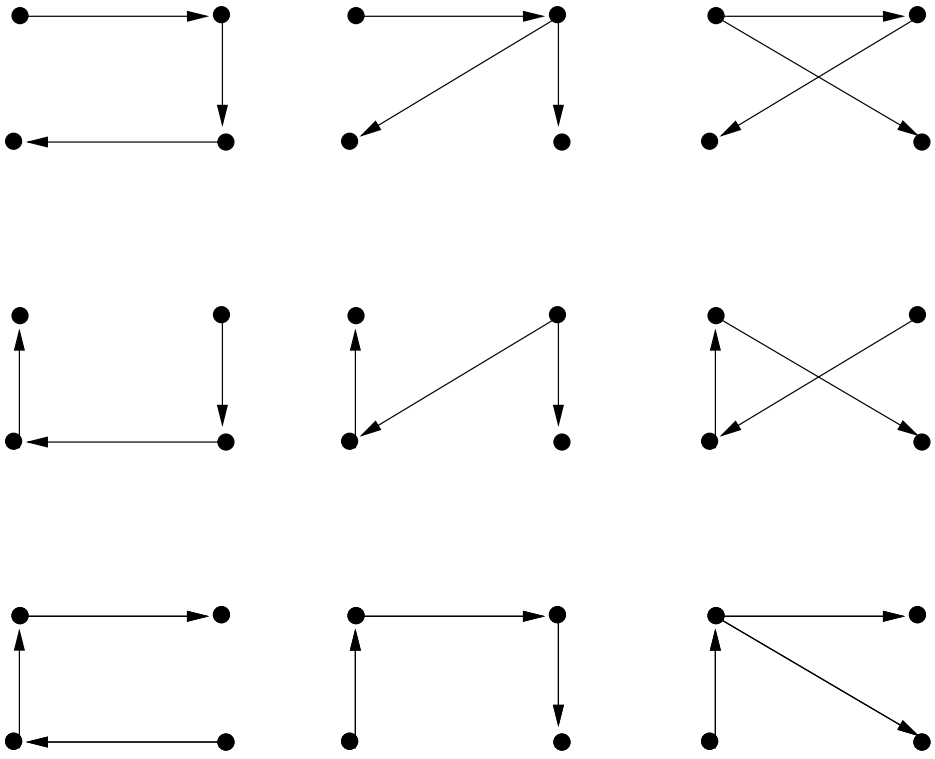


6. Draw all the spanning trees of the following directed graph:



SOLUTION

The answer is organized according to which node is chosen to be the root



SUPPLEMENTAL PROBLEM. **The Liar Problem [Smullyan]**

You are on an island hopping Carribean cruise. Each day your cruise ship docks at a different island where you spend the day touring. At one stop the *local culture* pamphlet explains that the inhabitants descend from two different tribes. Members of one tribe *always* tell the truth. Members of the other tribe *always* lie. Tribes members can be distinguished by the distinctive clothing they wear, but the pamphlet doesn't describe the difference.

After a day of sight-seeing, you loose track of time and discover you are late for your ship's departure! So you are in a real hurry to get back to the dock. Unfortunately, you become disoriented. Reaching a fork in the road, you do not know whether to go left or right.

Two inhabitants are standing by the road. They are differently dressed, so you know that one is from the truth-telling tribe and the other is from the tribe of liars. However, you don't know which is which. In the distance, you hear your ship sounding its horn. It's just about to leave! There is only time to ask one of the inhabitants a single question.

QUESTION: *Can you find out which way to go to reach the dock most quickly?*

COMMENT: Aren't you glad to be thinking about this now instead of when it actually happens? Keep track of the time you spend on this problem.

SOLUTION

Because they are differently dressed, one inhabitant must be a truth-teller and the other a liar. Ask either one, "Which way will your companion say is the dock?" Suppose the dock is to the left*. If you happen to ask a truth-teller, the answer will be "Right." If you happen to ask a liar, the answer will be "Right," as well. So no matter whom you ask, if the answer is "Right," the dock is to the left, and if the answer is "Left," the dock is to the right.*