

C241 Homework Assignment 2

1. (Ex. 2.1-1) Let $A = \{1, 2\}$ and $B = \{2, 3, 4\}$. Which of the following relations from A to B are functions?
- (a) $\{(1, 3), (2, 4)\}$ (d) $\{(1, 3), (2, 5)\}$
(b) $\{(1, 3), (1, 4)\}$ (e) $\{(2, 2), (1, 4)\}$
(c) $\{(1, 3), (1, 3)\}$

SOLUTION

- (a) $\{(1, 3), (2, 4)\}$ is a function from A into B .
- (b) $\{(1, 3), (1, 4)\}$ is not a function because there are two range values associated domain element 1.
- (c) $\{(1, 3), (1, 3)\}$ is not a function because there is no value associated with domain element 2.
- (d) $\{(1, 3), (2, 5)\}$ is not a function from A to B because $5 \notin B$.
- (e) $\{(2, 2), (1, 4)\}$ is a function from A into B .

2. (Ex. 2.1-2) Is $\{(1, 2), (2, 3)\}$ a function

- (a) from $\{(1, 2)\}$ to $\{(2, 3)\}$?
- (b) from \mathbb{N} to \mathbb{N} ?
- (c) from $\{1, 2\}$ to \mathbb{N} ?
- (d) from $\{1, 2, 3\}$ to $\{2, 3\}$?
- (e) from $\{1, 2, 3\}$ to $\{1, 2, 3\}$?

SOLUTION

- (a) *No; the domain and range elements are not ordered pairs.*
- (b) *No; but it is a partial function.*
- (c) *Yes, each element of the domain is related to exactly one element in the range.*
- (d) *No; but it is a partial function.*
- (e) *No; but it is a partial function.*

3. (Ex. 2.1-3) Let $A = \{1, 2\}$ and $B = \{2, 3, 4\}$.

- (a) List a relation that is an injective function from A to B .
- (b) List a relation that is a surjective function from B to A .
- (c) List two bijections from B to B .

SOLUTION

- (a) $\{(1, 2), (2, 3)\}$ is one example.
- (b) $\{(2, 1), (3, 2), (4, 2)\}$ is one example.
- (c) $\{(2, 2), (3, 3), (4, 4)\}$ and $\{(2, 3), (3, 4), (4, 2)\}$ are two possibilities.

4. (Ex. 2.1-8) Prove: If $f: A \rightarrow B$ and $g: B \rightarrow C$ are surjections (“onto functions”), then $g \circ f$ is onto.

SOLUTION

COMMENT. The proof narrative below describes the logic of the argument as it goes along and includes all the necessary steps and justifications. A more succinct argument would omit many of these details.

PROOF: *By Definition 2.6, $g \circ f: A \rightarrow C$ is surjective if for any element $c \in C$ there exists an $a \in A$ such that $(g \circ f)(a) = g(f(a)) = c$. Once $c \in C$ has been chosen, we must find a . Let $c \in C$. Since g is assumed to be surjective, By Definition 2.6 there is some $b \in B$ for which*

$$g(b) = c \tag{*}$$

Since f is assumed to be surjective, there must be some $a \in A$ for which $f(a) = b$. Hence, in equation () we can substitute*

$$g(f(a)) = (g \circ f)(a) = c$$

We have shown that for any $c \in C$ an $a \in A$ can be found for which $(g \circ f)(a) = c$. Thus, $g \circ f$ is surjective by Definition 2.6. \square

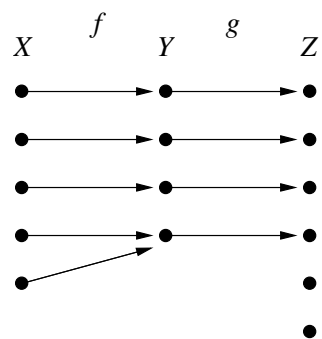
A briefer narrative would be sufficient.

PROOF: *Let $c \in C$. Since g is surjective, there is some $b \in B$ for which $g(b) = c$. Since f is also surjective, there exists an $a \in A$ for which $f(a) = b$. Thus, $(g \circ f)(a) = g(f(a)) = g(b) = c$. The choice of c was arbitrary; hence, $g \circ f$ is surjective by Definition 2.6. \square*

5. (Ex. 2.1-8) If $f: X \rightarrow Y$ is a surjection (onto) and $g: Y \rightarrow Z$ is an injection (one-to-one). Is $g \circ f$ always a surjection? Is it always an injection? If “yes” give a proof; and if “no” provide a counterexample.

SOLUTION

The composition of a surjection f , and an injection g , is not necessarily a surjection nor is it necessarily an injection. In the counterexample below $g \circ f$ is neither.



SUPPLEMENTAL PROBLEM. **The Farmer's Dilema**

A farmer is taking a head of cabbage and a goat to market. He brings along his faithful dog for company. On the road to the market there is a swift river that must be crossed. A small boat is tied to a tree on the bank of the river for this purpose. The boat is so small that it can carry the farmer and *just one* other thing, the cabbage, the goat or the dog. This is a problem: the farmer knows that, if left alone with the cabbage, the goat will eat it. And if left alone with the goat, the dog will eat it. Can the farmer safely get everything across the river? Or does he have to return home and leave the dog behind?

SOLUTION

- *The farmer takes the goat across the river, leaving the dog and the cabbage behind. The dog does not eat cabbage.*
- *Leaving the goat behind, the farmer returns to the other bank.*
- *The farmer takes the dog across the river.*
- *The farmer returns with the goat. Had he left the goat, the dog would eat it.*
- *The farmer leaves the goat, and takes the cabbage across.*
- *The farmer returns alone, leaving both the dog and the cabbage on the far bank of the river. Again, the dog won't eat the cabbage.*
- *The farmer retrieves the goat and makes the last trip across the river to the far bank.*
- *Now the dog, the goat, and the cabbage are on the far bank; and the farmer can resume the journey to the market*