C241 Homework Assignment 1

1. (Exercise 1.1-1) List the following sets:

(a) $\{2^i \mid i \in \mathbb{N} \text{ and } 0 \leq i \leq 8\}$

(b) $\{i^2 \mid i \in \mathbb{N} \text{ and } 0 \leq i \leq 8\}$

(c) $\{2k + 1 \mid k \in \mathbb{N}\}$

(d) $\{m \mid 23 < m < 29 \text{ and } m \text{ is a prime number}\}$

Solution

(a) {*1, 2, 4, 8, 16, 32, 64, 128, 256*}

(b) {*0, 1, 4, 9, 16, 25, 36, 49, 64*}

(c) {*1, 3, 5, 7, 9, ...*}

(d) $\emptyset$

2. (Exercise 1.1-2) Let $A = \{a, b\}$; let $B = \{1, 2, 3\}$; let $C = \emptyset$; and let $D = \{a, b, c, d\}$. List the following sets:

(a)	$A \cup B$	(f)	$A \cup C$
(b)	$A \cap B$	(g)	$A \cap D$
(c)	$A \times B$	(h)	A^3
(d)	$\mathcal{P}(A)$	(i)	$\mathcal{P}(\emptyset)$
(e)	$B \times \emptyset$	(j)	$(D \cap A) \times B$

Solution

(a) $\{a,\ b,\ 1,\ 2,\ 3\}$

(b) $\emptyset$

(c) $\{(a,\ 1),\ (a,\ 2),\ (a,\ 3),\ (b,\ 1),\ (b,\ 2),\ (b,\ 3)\}$

(d) $\{\emptyset,\ \{a\},\ \{b\},\ \{a,\ b\}\}$

(e) $\emptyset$

(f) $\{a,\ b\}$

(g) $\{a,\ b\}$

(h) $\{(a,\ a,\ a),\ (a,\ a,\ b),\ (a,\ b,\ a),\ (a,\ b,\ b),\ (b,\ a,\ a),\ (b,\ a,\ b),\ (b,\ b,\ a),\ (b,\ b,\ b)\}$

(i) $\{\emptyset\}$

(j) $\{(a,\ 1),\ (a,\ 2),\ (a,\ 3),\ (b,\ 1),\ (b,\ 2),\ (b,\ 3)\}$

3. Let $A = \{a, b\}$, $B = \{1, 3, 5\}$, and $C = \{\oplus, \otimes\}$.

(a) List the set $A \times B$.

(b) List the set $A \times (B \times C)$.

(c) List the set $A \times B \times C$.

Solution

(a) $\{(a, 1), (a, 3), (a, 5), (b, 1), (b, 2), (b, 3)\}$

(b) $B \times C = \{(1, \oplus), (3, \oplus), (5, \oplus), (1, \otimes), (3, \otimes), (5, \otimes)\}$, *so*

$$\begin{aligned} A\times(B\times C) = \; & \{\, (a,(1,\oplus)), (a,(3,\oplus)), (a,(5,\oplus)), (a,(1,\otimes)), (a,(3,\otimes)), (a,(5,\otimes)) \\ & (b,(1,\oplus)), (b,(3,\oplus)), (b,(5,\oplus)), (b,(1,\otimes)), (b,(3,\otimes)), (b,(5,\otimes)) \,\} \end{aligned}$$

(c) $A \times (B \times C)$ *is a set of ordered pairs whose second entry is also an ordered pair.* $A \times B \times C$ *is a set of* ordered triples

$$\begin{aligned} A \times B \times C = \; & \{\, (a,1,\oplus), (a,3,\oplus), (a,5,\oplus), (a,1,\otimes), (a,3,\otimes), (a,5,\otimes) \\ & (b,1,\oplus), (b,3,\oplus), (b,5,\oplus), (b,1,\otimes), (b,3,\otimes), (b,5,\otimes) \,\} \end{aligned}$$

4. (Exercise 1.1-3) Let $A = \{a, b\}$; let $B = \{1, 2, 3\}$; and let $E = A \times B$. List the following sets:

(a) $\{(x, y, y) \mid (x, y) \in E\}$

(b) $\{(x, x) \mid x \in E\}$

(c) $\{(y, z) \mid (x, y) \in E \text{ and } z \in B\}$

Solution

(a) {*(a, 1, 1), (a, 2, 2), (a, 3, 3), (b, 1, 1), (b, 2, 2), (b, 3, 3)*}

(b) {*((a, 1), (a, 1)), ((a, 2), (a, 2)), ((a, 3), (a, 3)), ((b, 1), (b, 1)), ((b, 2), (b, 2)), ((b, 3), (b, 3))*}

(c) {*(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)*}

5. List the first twelve elements of the set

$$F = \{f_k \in \mathbb{N} \mid f_1 = 1,\ f_2 = 2,\ \text{and for any } k \geq 1,\ f_{k+2} = f_k + f_{k+1}\}$$

That is, list $\{f_1, f_2, \ldots, f_{12}\}$

Solution

Elements f_1 and f_2 are given in the description,

$$F = \{1, 2, f_3, f_4, \ldots\}$$

The description specifies a formula for the rest of the f_is. $f_3 = f_1 + f_2 = 1+2 = 3$, so

$$F = \{1, 1, 2, f_4, f_5, \ldots\}$$

and similarly $f_4 = f_2 + f_3 = 2 + 3 = 5$, and so on.

$$F = \{1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, \ldots\}$$

6. Let $V = \{1, 2, 3\}$. List the set $\{xyxz | x, y, z \in V\}$ of words in V^+.

SOLUTION

$$\{1111, 1112, 1113, 1211, 1212, 1213, 1311, 1312, 1313,\\ 2121, 2122, 2123, 2221, 2222, 2223, 2321, 2322, 2323,\\ 3131, 3132, 3133, 3231, 3232, 3233, 3331, 3332, 3333\}$$

7. (Exercise 1.3-1) Let $V = \{\texttt{a}, \texttt{b}, \texttt{\$}\}$. For each of the following languages $L_i \subseteq V^+$, list enough elements to make it clear what each contains.

(a) In language L_1 each word has exactly one `$` and equally many `a` s as `b` s.

(b) In each word of language L_2, `a` s and `b` s alternate with any number of `$` s mixed in.

(c) In each word of language L_3, no `a` occurs next to a `b` .

(d) $L_4 = \{u \hat{} \texttt{\$} \hat{} v \mid u \in \{\texttt{a}\}^+ \text{ and } v \in \{\texttt{\$}, \texttt{b}\}^+\}$

(e) $L_5 = \{\texttt{a}^k \hat{} \texttt{\$} \hat{} \texttt{b}^k \mid k \in \mathbb{N}\}$

Solution

In most of these problems, it is hard to write down a pattern that everyone would recognize as describing the language in question.

(a) $L_1 = \{$`$ , $ a b ,$ b a , a $ b ,b $ a , a b $ ,b a $ ,`
`$ a a b b ,$ a b a b ,$ a b b a ,$ b a a b ,$ b a b a ,$ b b a a ,`
`a $ a b b ,a $ b a b ,a $ b b a ,b $ a a b ,b $ a b a ,b $ b a a ,`
`a a $ b b ,a b $ a b ,a b $ b a ,b a $ a b ,b a $ b a ,b b $ a a ,`
`a a b $ b ,a b a $ b ,a b b $ a ,b a a $ b ,b a b $ a ,b b a $ a ,`
`a a b b $ ,a b a b $ ,a b b a $ ,b a a b $ ,b a b a $ ,b b a a $ ,`
`$ a a a b b b` $, \ldots\}$

(b) $L_2 = \{\varepsilon,$ `$ , $ $ , $ $ $` $, \ldots$
`a , $ a , a $ , $ $ a , $ a $ , a $ $ , $ $ $ a , $ $ a $` $, \ldots$
`b , $ b , b $ , $ $ b , $ b $ , b $ $ , $ $ $ b , $ $ b $` $, \ldots$
`a b , $ a b , a $ b , a b $ , $ $ a b , $ a $ b , $ a b $ , a $ b $ , a b $ $`
`$ , $ $ $ a b` $, \ldots$
`b a , $ b a , b $ a , b a $ , $ $ b a , $ b $ a , $ b a $ , b $ a $ , b a $ $`
`$ , $ $ $ b a` $, \ldots$
`a b a , $ a b a , a $ b a` $, \ldots\}$

(c) $L_3 = \{\varepsilon,$`a ,b ,$  a $ ,$ a ,b $ ,$ b ,$ $ , a a $ ,a $ a ,$ a a ,b b $ ,b $ b`
`,$ b b ,$ $ a ,$ $ b ,$ $ $`
`a a a $ ,a $ a a ,`$\ldots\}$

(d) $L_4 = \{\varepsilon,$`a $ $ ,a $ b ,`
`a $ $ $ ,a $ $ b ,a $ b $ ,a $ b b ,`
`a $ $ $ $ ,a $ $ $ b ,a $ $ b $ ,a $ $ b b ,a $ b $ $ ,a $ b $ b ,a $ b $ b`
`,a $ b b $ ,a $ b b b ,`
$\ldots,$ `a a $ $ ,a a $ $ b ,a a $ b $ ,a a $ b b ,`
`a a $ $ $ ,a a $ $ $ b ,a a $ $ b $ ,a a $ $ b b ,a a $ b $ $ ,a a $ b $`
`b ,a a $ b $ b ,a a $ b b $ ,a a $ b b b ,`
$\ldots,\}$

(e) $L_5 = \{$`$ ,a $ b ,a a $ b b ,a a a $ b b b ,a a a a $ b b b b ,`$\ldots\}$

8.

What does this STMT program compute? Trace its execution by hand for a few small values or A and B. Place a statement in the empty assertion at the end of the program saying what condition holds at that point. Explain why the program satisfies your assertion when it reaches the `end` .

```
{x = A ∈ 𝕎 and y = B ∈ 𝕎}
begin
while  x ≠ y do
   if  x  <  y
     then  y := y − x
     else  x := x − y
end
```

{ [ ] }

(Optional) Write and test a program in the language of your choice (preferably Scheme) that performs the same computation as the given program above.

Solution

Label the program statements as follows.

```
    { x = A ∈ 𝕎 and y = B ∈ 𝕎 }
    begin
1:  while  x ≠ y do
2:     if  x  <  y
3:       then  y := y − x
4:       else  x := x − y
5:  end
```

Below, the program is traced for a few small values of A and B.

$A = 6,\ B = 9$

pc	x	y
1	6	9
3	6	3
1	6	3
4	3	3
5	3	3

$A = 6, B = 15$

pc	x	y
1	6	15
3	6	9
1	6	9
3	6	3
1	6	3
4	3	3
5	3	3

$A = 35, B = 14$

pc	x	y
1	35	14
4	21	14
1	21	14
4	7	14
1	7	14
3	7	7
5	7	7

The postcondition would be something like

{ *x is the greatest common divisor of A and B* }

Supplemental Problem. A small island is ruled by a benevolent queen. One morning, a proclamation is posted in the central common:

> **—To All My Subjects—**
>
> It has come to my attention that one or more husbands in my realm are unfaithful to their wives. I hereby decree that any woman who learns her husband is unfaithful must shoot him at the stroke of midnight.
>
> *Her Highness, The QUEEN*

It's a small island. Every wife knows who all the unfaithful husbands are, *but does not know whether or not her <u>own</u> husband is faithful.*

On the third night, shots ring out. How many husbands are/were unfaithful?

Solution

There were three unfaithful husbands, now all deceased.
To see why this is so, consider what would happen if there were:

1. just one unfaithful husband: *His wife knows:*
   - *that all of all husbands except possibly her own are faithful.*
   - *that there is at least one unfaithful husband because the Queen's proclamation says so.*

   She must conclude her own husband is unfaithful (in fact, is the only unfaithful one), so at midnight of the first night, she would shoot him.

2. exactly two unfaithful husbands. *Their wives* both *know:*
   - *There is* at least one *unfaithful husband, the one each knows about.*
   - *There are* at most two *unfaithful husbands, if the wife's own husband is unfaithful.*
   - *There are* more than one *unfaithful husbands, because no shots are heard on the first night.*

   Both wives must conclude that their own husbands are unfaithful (and, in fact, that there are exactly two unfaithful husbands), so at midnight of the second night they shoot them.

n. *In general, if a wife knows of n unfaithful husbands, she must wait until the n^{th} night to determine whether her own husband is unfaithful. If shots are heard, he is not; if no shots are heard, he is (and, in fact, there are exactly $n+1$ unfaithful husbands in all). So at midnight on the $(n+1)^{\text{st}}$ night, each of these $n+1$ wives will shoot her husband.*