c241 Self Test

Instructions:

- *Finish the test before looking at the answers.*
- *Give yourself 60 minutes to answer the Questions. If you cannot answer a question completely, write what you can. You should be able answer four of these questions (or come close).*
- *Read the solutions and compare with your answers. You should be able to read and understand all the solutions. The technique used to solve the problem is more important than whether the final answer, is correct.*

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1. Without using a calculator, perform fractional arithmetic below, reducing your answer to lowest terms.

   (a) $\frac{420}{147} + \frac{35}{20} = \boxed{\frac{N}{M}?}$

   (b) $\frac{35}{726} \div \frac{20}{154} = \boxed{\frac{N}{M}?}$

Solution

The easiest approach is to reduce the numbers to their prime factors first, for example,

$$420 = 42 \cdot 10 = (7 \cdot 6) \cdot 10 = 7 \cdot (3 \cdot 2) \cdot (5 \cdot 2) = 2 \cdot 2 \cdot 3 \cdot 5 \cdot 7$$

$$
\begin{aligned}
(a) &\quad \frac{420}{147} + \frac{35}{20} \\
= &\quad \frac{2\cdot 2\cdot 3\cdot 5\cdot 7}{3\cdot 7\cdot 7} + \frac{5\cdot 7}{2\cdot 2\cdot 5} \\
= &\quad \frac{2\cdot 2\cdot 3\cdot 5\cdot 7\cdot (2\cdot 2\cdot 5)}{2\cdot 2\cdot 3\cdot 5\cdot 7\cdot 7} + \frac{5\cdot 7\cdot (3\cdot 7\cdot 7)}{2\cdot 2\cdot 3\cdot 5\cdot 7\cdot 7} \\
= &\quad \frac{2\cdot 2\cdot 2\cdot 2\cdot 3\cdot 5\cdot 5\cdot 7}{2\cdot 2\cdot 3\cdot 5\cdot 7\cdot 7} + \frac{3\cdot 5\cdot 7\cdot 7\cdot 7}{2\cdot 2\cdot 3\cdot 5\cdot 7\cdot 7} \\
= &\quad \frac{\cancel{3}\cdot \cancel{5}\cdot \cancel{7}\cdot (2\cdot 2\cdot 2\cdot 2\cdot 5 + 7\cdot 7)}{2\cdot 2\cdot \cancel{3}\cdot \cancel{5}\cdot \cancel{7}\cdot 7} \\
= &\quad \frac{80+49}{28} \\
= &\quad \frac{129}{48}
\end{aligned}
$$

$$
\begin{aligned}
(b) &\quad \frac{35}{726} \div \frac{20}{154} \\
= &\quad \frac{5\cdot 7}{2\cdot 3\cdot 11\cdot 11} \div \frac{2\cdot 2\cdot 5}{2\cdot 7\cdot 11} \\
= &\quad \frac{5\cdot 7}{2\cdot 3\cdot 11\cdot 11} \times \frac{2\cdot 7\cdot 11}{2\cdot 2\cdot 5} \\
= &\quad \frac{(5\cdot 7)\cdot (2\cdot 7\cdot 11)}{(2\cdot 3\cdot 11\cdot 11)\cdot (2\cdot 2\cdot 5)} \\
= &\quad \frac{\cancel{2}\cdot \cancel{5}\cdot 7\cdot 7\cdot \cancel{11}}{\cancel{2}\cdot 2\cdot 2\cdot 3\cdot \cancel{5}\cdot 11\cdot \cancel{11}} \\
= &\quad \frac{7\cdot 7}{2\cdot 2\cdot 3\cdot 11} \\
= &\quad \frac{49}{132}
\end{aligned}
$$

2. Is there any logical difference between the following two sentences?

   (a) "If P then if Q then R."

   (b) "If P and Q then R."

Solution

Statements A and B are equivalent. This can be shown...

(a) *with a truth table:*

P	Q	R	P	$\Rightarrow$	$(Q$	$\Rightarrow$	$R)$	$(P$	$\wedge$	$Q)$	$\Rightarrow$	R
F	F	F	F	**T**		T			F		**T**	F
F	F	T	F	**T**		T			F		**T**	T
F	T	F	F	**T**		F			F		**T**	F
F	T	T	F	**T**		T			F		**T**	T
T	F	F	T	**T**		T			F		**T**	F
T	F	T	T	**T**		T			F		**T**	T
T	T	F	T	**F**		F			T		**F**	F
T	T	T	T	**T**		T			T		**T**	T

(b) *or an argument by cases, such as*

- *If P is true, the both statements reduce to "If Q the R."*
- *If P is false, then Statement (a) is true and, because "P and Q" is false, so is Statement B.*

3. Prove that $\dfrac{k(k+1)}{2} + k + 1 = \dfrac{(k+1)(k+2)}{2}$

SOLUTION

$$
\begin{aligned}
&\frac{k(k+1)}{2} + k + 1 \\
&\quad = \quad \frac{k(k+1)}{2} + \frac{2(k+1)}{2} && \textit{(multiply } k+1 \textit{ by } \tfrac{2}{2}\textit{)} \\
&\quad = \quad \frac{k^2 + k + 2k + 2}{2} && \textit{(add fractions)} \\
&\quad = \quad \frac{k^2 + 3k + 2}{2} && \textit{(simplify the numerator)} \\
&\quad = \quad \frac{(k+1)(k+2)}{2} && \textit{(factor the numerator)}
\end{aligned}
$$

4. If you flip a coin, the outcome is either “heads” (H) or “tails” (T). Suppose you flip a coin seven times. One possibility is $H\ H\ H\ H\ H\ H\ H$; that is, all seven flips result in “heads.” How many possible outcomes are there in all?

   Solution

   There are two possible outcomes for each flip, so the number of possible outcomes is $2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 2^7 = 128$.

5. Simplify $\dfrac{6x^4 + 10x^3 + 26x^2 + 20x + 28}{2x^2 + 4}$

Solution

$$
\begin{array}{rrrrrr|rrrrrrrrr}
 & & & & & & & & & & & 3x^2 & + & 5x & + & 7 \\
\cline{7-16}
2x^2 & + & 0x & + & 4 & & & 6x^4 & + & 10x^3 & + & 26x^2 & + & 20x & + & 28 \\
 & & & & & & - & 6x^4 & + & 0x^3 & + & 12x^2 & & & & \\
\cline{8-16}
 & & & & & & & & = & 10x^3 & + & 14x^2 & + & 20x & & \\
 & & & & & & & & - & 10x^3 & + & 0x^2 & + & 20x & & \\
\cline{10-16}
 & & & & & & & & & & = & 14x^2 & + & 0x & + & 28 \\
 & & & & & & & & & & - & 14x^2 & + & 0x & + & 28 \\
\cline{12-16}
 & & & & & & & & & & & & & & = & 0
\end{array}
$$

6. A cafeteria offers a choice of 3 appetizers, 3 salads, 2 soups, 5 entrees, 2 desserts, and 4 drinks.

   (a) A *full meal* consists of an appetizer, either soup or a salad but not both, an entree, a dessert, and a drink. How many different full meals are there?

   (b) A *quick meal* consists of either an appetizer or a soup and a salad but not both, an entree, and a drink. How many different quick meals are there?

Solution

(a) *The number of different full meals is* $3 \cdot (3 + 2) \cdot 5 \cdot 2 \cdot 4 = 600$.

(b) *The number of quick meals is* $(3 + (3 \cdot 2)) \cdot 5 \cdot 4 = 360$.

7. For what value or values of x is $2x^2 + x - 6 = 0$?

SOLUTION

(a) *Try factoring the quadratic. Find* m *and* n *that satisfy* $mn = -6$ *and* $2n + m = 1$. $m = 3$ *and* $n = -2$ *work:* $2x^2 + x - 6 = (2x - 3)(x + 2)$. *This product is* 0 *when* $2x = 3$ *or* $x = -2$, *so the roots of the quadratic are* $\frac{3}{2}$ *and* -2.

(b) *Alternatively, use the* quadratic equation

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

and compute

$$\frac{-1 \pm \sqrt{1^2 - (4 \cdot 2 \cdot (-6))}}{2 \cdot 2} = \frac{-1 \pm \sqrt{49}}{4} = \frac{\pm 7 - 1}{4} = \begin{cases} \frac{6}{4} = \frac{3}{2} \\ \\ \frac{-8}{4} = -2 \end{cases}$$

8. Andy, Bob, Cindy, Dinah, Eve, Fred, and Gary live in the seven houses, numbered 1 through 7, on Maple Steet. Gary's address is 5 greater than Bob's. Bob's address is greater than Andy's. Dinah's address is less than Eve's, whose address is 2 less than Gary's. Cindy's address is less than either Dinah's or Fred's. Who lives where?

SOLUTION

One way to solve logic problems of this kind is by a process of elimination. Draw a matrix of addresses and people. Let the variables A, B, etc. stand for Andy's address, Bob's address, etc.

	1	2	3	4	5	6	7
A							
B							
C							
D							
E							
F							
G							

We procede by marking the choices that are eliminated by the constraints expressed in the problem statement. The exposition that follows looks long, but bear in mind that all the steps are performed on a single *matrix, which accumulates information as we go along.*

1. Gary's address is 5 greater than Bob's. *This clue can be written $G = B + 5$. Since we also know that $G \leq 7$, it must be the case that $B \leq 2$, and $G \geq 6$. This is all the information we can infer from the first clue.*

	1	2	3	4	5	6	7
A							
B			•	•	•	•	•
C							
D							
E							
F							
G	•	•	•	•	•		

2. Bob's address is greater than Andy's, *that is, $B > A$. The only possibility is $A = 1$ and $B = 2$. So we can mark out all the other possibilities in the first two rows. Clue 1 says $B + 5 = G$, so $G = 7$. Once we know who has a given address, we can mark out all the other possibilities in its column.*

	1	2	3	4	5	6	7
A	$\surd$	•	•	•	•	•	•
B	•	$\surd$	•	•	•	•	•
C	•	•					•
D	•	•					•
E	•	•					•
F	•	•					•
G	•	•	•	•	•	•	$\surd$

3. Dinah's address is less than Eve's, whose address is 2 less than Gary's. *This clue gives us two pieces of information, $D < E$ and $E + 2 = G$. Since $G = 7$, $E = G - 2 = 5$ leaving two possibilities for D, $D = 3$ or $D = 4$. We could guess and see what happens, but instead, let's see*

what information the last clue offers.

	1	2	3	4	5	6	7
A	√	•	•	•	•	•	•
B	•	√	•	•	•	•	•
C	•	•			•		•
D	•	•			•		•
E	•	•	•	•	√	•	•
F	•	•			•		•
G	•	•	•	•	•	•	√

4. Cindy's address is less than either Dinah's or Fred's. *There are only three addresses left open. This clue says $C < D$ and $C < F$, that is, C is the smallest. So it must be that $C = 3$. Clue 3 says $D < E = 5$, and the only remaining address that satisfies that constraint is $D = 4$, which leaves $F = 5$ as the only choice of F.*

	1	2	3	4	5	6	7
A	√	•	•	•	•	•	•
B	•	√	•	•	•	•	•
C	•	•	√	•	•	•	•
D	•	•	•	√	•	•	•
E	•	•	•	•	√	•	•
F	•	•	•	•	•	√	•
G	•	•	•	•	•	•	√

Now everybody has an address, but as always, we should check our solution, by verifying that all the clues are satisfied.

(a) Is Gary's address is 5 greater than Bob's? *Yes:* $2 + B = 2 + 5 = 7 = B$.

(b) Is Bob's address is greater than Andy's? *Yes:* $B = 2 > 1 = A$.

(c) Is Dinah's address less than Eve's? *Yes:* $D = 4 < 5 = E$.

(d) Is Eve's address 2 less than Gary's? *Yes:* $E = 5 = 7 - 2 = G - 2$.

(e) Is Cindy's address less than either Dinah's or Fred's? *Yes:* $C = 3 < 4 = D$ *and* $C = 3 < 5 = E$.