NAME: ____________________

ID: ____________________

C241 Test Two

1	/12
2	/12
3	/12
4	/12
5	/12
6	/12
7	/12
8	/12
	/96

INSTRUCTIONS:

- *Put your name and ID number in the upper-right of each page.*
- *Exam time is 75 minutes.*
- *There eight equally weighted questions.*
- *Place your final answer on the test in the space provided. Scratch work is not graded, but neatness counts. Mark the parts of your answers clearly.*
- *If you see a mistake or are uncertain about the meaning of a problem, clearly state write your interpreting, then go on to answer accordingly.*

1. (12 points) Define $x \oplus y$ to be $x\overline{y} + \overline{x}y$. Use the boolean identities to prove $\overline{(x \oplus y)} = \overline{x} \oplus y$

SOLUTION

	$\overline{(x \oplus y)}$	
$=$	$\overline{\overline{x}\,y + x\,\overline{y}}$	*Defn. '$\oplus$'*
$=$	$\overline{\overline{x}\,y}\,\overline{x\,\overline{y}}$	*DeMorgan's Law*
$=$	$(\overline{\overline{x}} + \overline{y})\,(\overline{x} + \overline{\overline{y}})$	*DeMorgan's Law*
$=$	$\overline{\overline{x}}\,\overline{x} + \overline{\overline{x}}\,\overline{\overline{y}} + \overline{y}\,\overline{x} + \overline{y}\,\overline{\overline{y}}$	*Distributivity*
$=$	$x\,\overline{x} + \overline{\overline{x}}\,y + \overline{y}\,\overline{x} + \overline{y}\,y$	*Negation*
$=$	$0 + \overline{\overline{x}}\,y + \overline{y}\,\overline{x} + 0$	*Cancellation, twice*
$=$	$\overline{\overline{x}}\,y + \overline{y}\,\overline{x}$	*Identity, twice*
$=$	$\overline{\overline{x}}\,y + \overline{x}\,\overline{y}$	*Commutativity*
$=$	$\overline{x} \oplus y$	*Defn. '$\oplus$'*

2. (12 points) Determine whether the following proposition is a tautology.

$$(a \vee b \Leftrightarrow c) \wedge (d \vee e) \Leftrightarrow ((a \vee b \Leftrightarrow c) \wedge d) \vee ((a \vee b \Leftrightarrow c) \wedge e)$$

SOLUTION

The problem is simpler to solve if you recognize that a single term occurs three times in the proposition:

$$\forall \wedge (d \vee e) \Leftrightarrow (\forall \wedge d) \vee (\forall \wedge e)$$

Call this repeated term X. It should suffice to determine whether

$$X \wedge (d \vee e) \Leftrightarrow (X \wedge d) \vee (X \wedge e)$$

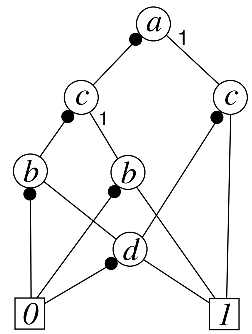
is a tautology because the orginal proposiiton can be obtained by substituting term $\forall$ for the X. The truth table for this term has eight cases, where the original had thirty-two.

X	d	e	X	$\wedge$	$(d \vee e)$	$\Leftrightarrow$	$(X \wedge d)$	$\vee$	$(X \wedge e)$
F	F	F	F	F	F	T	F	F	F
F	F	T	F	F	T	T	F	F	F
F	T	F	F	F	T	T	F	F	F
F	T	T	F	F	T	T	F	F	F
T	F	F	T	F	F	T	F	F	F
T	F	T	T	T	T	T	F	T	T
T	T	F	T	T	T	T	T	T	F
T	T	T	T	T	T	T	T	T	T

Thus, the proposition is a tautology.

In general we should check whether $a \vee b \Leftrightarrow c$ is a tautology or a contradiction. In either case the problem is easier because the truth table is smaller, but it doesn't matter what X evaluates to; $X \wedge (d \vee e) \Leftrightarrow (X \wedge d) \vee (X \wedge e)$ still evaluates to T.

3. (12 points) The ROBDD to the right represents a term over variable ordering $\langle a,\ c,\ b,\ d\rangle$ Give the disjunctive normal form (DNF) for this term under the variable ordering $\langle a,\ b,\ c,\ d\rangle$.



SOLUTION

Summing the terms for every path to $\boxed{1}$, we have

$$\overline{a}\,\overline{c}\,b\,d + \overline{a}\,c\,b + a\,\overline{c}\,d + a\,c$$

Reordering the clauses yields

$$\overline{a}\,b\,\overline{c}\,d + \overline{a}\,b\,c + a\,\overline{c}\,d + a\,c$$

Introducing all variables to each clause gives

$$\overline{a}\,b\,\overline{c}\,d + (\overline{a}\,b\,c\,{\color{red}d} + \overline{a}\,b\,c\,{\color{red}\overline{d}}) + (a\,{\color{red}b}\,\overline{c}\,d + a\,{\color{red}\overline{b}}\,\overline{c}\,d) + (a\,{\color{red}b}\,c\,{\color{red}d} + a\,{\color{red}b}\,c\,{\color{red}\overline{d}} + a\,{\color{red}\overline{b}}\,c\,{\color{red}d} + a\,{\color{red}\overline{b}}\,c\,{\color{red}\overline{d}})$$

Rearranging clauses (not required but easier to check), we get

$$a\,b\,c\,d + a\,b\,c\,\overline{d} + a\,b\,\overline{c}\,d + a\,\overline{b}\,c\,d + a\,\overline{b}\,c\,\overline{d} + a\,\overline{b}\,\overline{c}\,d + \overline{a}\,b\,c\,d + \overline{a}\,b\,c\,\overline{d} + \overline{a}\,b\,\overline{c}\,d$$

4. (12 points) There are 60 Computer Science majors enrolled in at least one of C241, C343 or C335.

(a) One student is enrolled in all three classes.

(b) Four students are enrolled in C343 and C335.

(c) Four students are enrolled in C241 and C343.

(d) 32 students are enrolled in C241.

(e) 19 students are enrolled in C343.

(f) 22 students are enrolled in C335.

How many students are enrolled in C241 and C335?

SOLUTION

Let $A = \{s \mid s \text{ is in } C241\}$, $B = \{s \mid s \text{ is in } C343\}$, and $C = \{s \mid s \text{ is in } C335\}$. The question is asking us to find the size of $A \cup C$. Recall[1] that

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

(The derivation is below.) The question gives us seven of these numbers, so we can solve for the eighth:

$$\begin{array}{cccccccccccccc} (1st\ sentence) & & (d) & & (e) & & (f) & & (c) & & & & (b) & & (a) \\ |A \cup B \cup C| & = & |A| & + & |B| & + & |C| & - & |A \cap B| & - & |A \cap C| & - & |B \cap C| & + & |A \cap B \cap C| \\ 60 & = & 32 & + & 19 & + & 22 & - & 4 & - & |A \cap C| & - & 4 & + & 1 \end{array}$$

Hence,

$$\boxed{|A \cap C|} = 32 + 19 + 22 - 4 - 4 + 1 - 60 \boxed{= 6}$$

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[1]Problem 7 of Homework Assignment 5. The derivation is based on the fact given in Section 4.1 that $|A \cup B| = |A| + |B| - |A \cap B|$. Extending to three sets and using some boolean algebra,

$$\begin{array}{ll} & |(A \cup B) \cup C| \\ = & |A \cup B| + |C| - |(A \cup B) \cap C| \\ = & (|A| + |B| - |A \cap B|) + |C| - |(A \cup B) \cap C| \\ = & |A| + |B| + |C| - |A \cap B| - |(A \cup B) \cap C| \\ = & |A| + |B| + |C| - |A \cap B| - |(A \cap C) \cup (B \cap C)| \\ = & |A| + |B| + |C| - |A \cap B| - (|A \cap C| + |B \cap C| - |(A \cap C) \cap (B \cap C)|) \\ = & |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |(A \cap C) \cap (B \cap C)| \\ \overset{\star}{=} & |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |(A \cap B \cap C)| \end{array}$$

5. (12 points) You have two wall shelves for displaying selected items from your collection of 99 beer bottles. Each shelf can hold as many as 5 bottles, but you decide to display exactly nine bottles.

(a) Assuming that shelf-order and bottle-order matters, how many different displays can you set up?

(b) Assuming that shelf-order doesn't matter, how many different displays can you set up?

(c) Assuming that bottle-order matters on the top shelf but not on the bottom shelf, how many different displays can you set up?

SOLUTION

First, since each shelf can hold no more than five bottles, and nine bottles are displayed, five of the bottles must go on one shelf, and four on the other.

(a) If self-order and bottle-order both matter: Choose a 9-permutation from 99 bottles; then **either** put the first five bottles on the top shelf and the remaining four on the bottom shelf **or** the first four on the top shelf and the last five on the bottom shelf. The result is

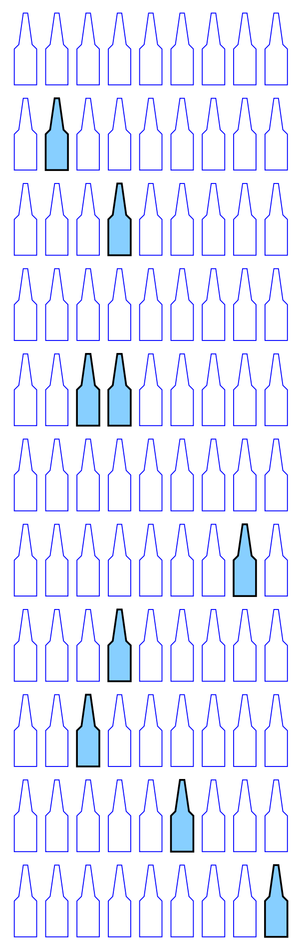
$$2 \cdot \frac{99!}{90!}$$

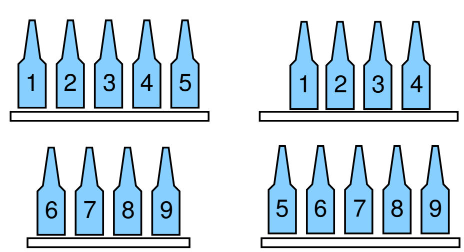
(b) If shelf-order doesn't matter then you don't have to decide which shelf gets five bottles. So the result is

$$\frac{99!}{90!}$$

(c) If bottle order counts on the top shelf but not the bottom then **either** choose a 5-permutation for the top shelf and choose an unorded subset of 4 bottles from the remaining 94 bottles in the collection **or** choose a 4-permutation for the top shelf and an unorderd subset of 5 bottles from the remaining 95 bottles. The resulting count is

$$\frac{99!}{94!} \cdot \binom{94}{4} + \frac{99!}{95!} \cdot \binom{95}{5}$$





6. (12 points)

(a) How many different permutations are there of the letters in the word BUBBLEGUM?

(b) how many different permutations are there if the letters G and M must *not* appear next to each other?

SOLUTION

(a) There are 9 letters in BUBBLEGUM but this includes 3 Bs and 2 Us. So the number of different permutations is

$$\frac{9!}{3! \cdot 2!}$$

(b) It is easier to count the number of permutations in which G and M *are* adjacent, then subtract that number from the total.

1. If G and M adjacent, they appear either as GM or MG.
2. Together they may be thought of as one letter, so the number of permutations is 8!
3. As before, divide out the redundant permutations of Bs and Us.
4. So the number of permutations in which G and M are adjacent is $2 \cdot \dfrac{8!}{3! \cdot 2!}$

The number of purmutations in which G and M are <u>not</u> adjacent, then, is

$$\frac{9!}{3! \cdot 2!} - \frac{2 \cdot 8!}{3! \cdot 2!}$$

The result can be *but is not required to be* simplified.

$$\begin{aligned} & \frac{9!}{3! \cdot 2!} - \frac{2 \cdot 8!}{3! \cdot 2!} \\ = \quad & \frac{9 \cdot 8!}{3! \cdot 2!} - \frac{2 \cdot 8!}{3! \cdot 2!} \\ = \quad & \frac{7 \cdot 8!}{3! \cdot 2!} \\ = \quad & \frac{7 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4}{2!} \\ = \quad & 7^2 \cdot 6 \cdot 5 \cdot 4^2 \\ = \quad & 23520 \end{aligned}$$

7. (12 points) Use induction to prove that the sum of the first n odd numbers is equal to n^2. That is, show:

For all $n \in Nat$, $n > 0$, $\sum_{i=1}^{n}(2i-1) = n^2$.

SOLUTION

Base Case. $\sum_{i=1}^{1}(2i-1) = 2 \cdot 1 - 1 = 1 = 1^2$

Induction. Assume that $\sum_{i=1}^{k}(2i-1) = k^2$.

$$\begin{aligned}
\sum_{i=1}^{k+1}(2i-1) &= 2(k+1) - 1 + \boxed{\sum_{i=1}^{k}(2i-1)} && \text{(expanding } \textstyle\sum \text{ by one term)} \\
&\overset{H}{=} 2(k+1) - 1 + \boxed{k^2} && \text{(I.H.)} \\
&= k^2 + 2k + 1 && \text{(simplifying)} \\
&= (k+1)^2 && \text{(factoring)}
\end{aligned}$$

8. (12 points)

The program below is claimed to computer the product of two natural numbers. According to the Fundamental Theorem on Invariants, there are three conditions that must be established to verify this program satisfies its final assertion. State what they are and explain (or prove) whether they are true.

```
P:
   {A ∈ ℕ ∧ B ∈ ℕ}
   begin
   x := A;
   y := B;
   z := 0;
   while y ≠ 0 {z + xy = A · B} do
     if even?(y)
     then
       begin y := y ÷ 2; x := 2 × x end
      else
       begin y := y − 1 ; z := z + x end
   end
   {z = A · B}
```

SOLUTION

(a) INITIALIZATION: *When $\mathcal{P}$ first reaches the* while *loop, $z + xy = A \cdot B$.*

$\mathcal{P}$ has just assigned A to x, B to y, and 0 to z, so $z + xy = 0 + A \cdot B = A \cdot B$.

(b) INVARIANCE: *If $x + xy = A \cdot B$ and $y \neq 0$ the body of the* while *loop makes $z + xy = A \cdot B$ true.*

There are two cases to consider, depending on the outcome of the test $even?(y)$.

CASE $even?(y)$. If y is even, the then branch computes new values $x' = 2x$, $y' = \frac{y}{2}$, and $z' = z$. Dividing y by 2 is legal, since in this case y is even, so the quotient is still in $\mathbb{N}$. The invariant is reestablished because $z' + x'y' = z + 2x \cdot \frac{y}{2} = z + xy$, and it is assumed that $z + xy = A \cdot B$.

CASE $\neg even?(y)$. If y is not even, the else branch computes new values $x' = x$, $y' = y - 1$, and $z' = z + x$. The invariant is reestablished because $z' + x'y' = (z + x) + x(y - 1) = z + xy + x - x = z + xy$, which is assumed equal to $A \cdot B$.

(c) TERMINATION: *On termination the final assertion is true.*

The Theorem on Invariance says that if INITIALIZATION and INVARIANCE hold, $z + xy = A \cdot B \wedge y = 0$ when the while loop terminates.. But then we have $z + xy = z + x \cdot 0 = z = A \cdot B$, as desired.

Reference *You may detach this page.*

The Boolean Identitities.

Negation	$\overline{\overline{x}} = x$					
Identity	$0 + x$	$=$	x	$1 \cdot x$	$=$	x
Dominance	$1 + x$	$=$	1	$0 \cdot x$	$=$	0
Idempotence	$x + x$	$=$	x	$x \cdot x$	$=$	x
Cancellation	$x + \overline{x}$	$=$	1	$x \cdot \overline{x}$	$=$	0
Commutativity	$x + y$	$=$	$y + x$	$x \cdot y$	$=$	$y \cdot x$
Associativity	$x + (y + z)$	$=$	$(x + y) + z$	$x \cdot (y \cdot z)$	$=$	$(x \cdot y) \cdot z$
Distributivity	$x \cdot (y + z)$	$=$	$(x \cdot y) + (x \cdot z)$	$x + (y \cdot z)$	$=$	$(x + y) \cdot (x + z)$
DeMorgan	$\overline{(x + y)}$	$=$	$\overline{x} \cdot \overline{y}$	$\overline{(x \cdot y)}$	$=$	$\overline{x} + \overline{y}$

Fact 4.1. For finite sets A and B

(a) $|A \cup B| = |A| + |B| - |A \cap B|$

(b) $|A \cap B| = |A| + |B| - |A \cup B|$

(c) $|A \times B| = |A| \cdot |B|$

(d) $|A \setminus B| = |A| - |A \cap B|$

Definition 4.3 There are $\dfrac{n!}{(n-m)!}$ *m-permutations on n elements.* This is the number of distinct ways to order m distinct elements from a set of size n.

Definition 4.4 The number of k-element subsets of a set of size n is $\dbinom{n}{k} = \dfrac{n!}{k! \cdot (n-k)!}$.

Theorem 5.1 (Theorem on Loop Invariants) In the STMT programming language let B be any test, S any statement, and consider the program fragment

$$\begin{array}{ll} \mathcal{P}\colon & \vdots \\ \ell\colon & \texttt{while}\ B\ \texttt{do}\ \{I\ \}\, S\,\texttt{;} \\ \ell'\colon & \vdots \end{array}$$

Suppose assertion I has the property that, whenever both I and B hold before S executes, I holds again after S executes. Then if $\mathcal{P}$ reachs ℓ with I true, when (and if) $\mathcal{P}$ reaches ℓ', I will be true and B will be false.