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C241 Test One

1	/12
2	/15
3	/ 8
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5	/15
6	/15
7	/10
8	/10
9	/10
	/95

INSTRUCTIONS:

- *Put your name and ID number in the upper-right of each page.*
- *Place your final answer on the test in the space provided. Scratch work is not graded, but neatness counts. Mark the parts of your answers clearly.*
- *Exam time is 75 minutes.*
- *There eight questions, weighted as indicated.*
- *If you find an error or ambiguity in a problem, describe it and state how you would correct it. Then go on to answer the question using your interpretation.*

1. (12 points) List the following sets:

(a) $\{2^i \mid i \in \mathbb{N} \text{ and } 0 \leq i \leq 8\} = \{1, 2, 4, 8, 16, 32, 64, 128, 256\}$

(b) $\{i^2 \mid i \in \mathbb{N} \text{ and } 0 \leq i \leq 8\} = \{0, 1, 4, 9, 16, 25, 36, 49, 64\}$

(c) $\{2k + 1 \mid k \in \mathbb{N}\} = \{1, 3, 5, 7, 9, \ldots\}$

(d) $\{m \mid 23 < m < 29 \text{ and } m \text{ is a prime number}\} = \emptyset$

2. (15 points) *Let* $A = \{1, 2, 3, 4, 5\}$ *and* $B = \{1, 3, 5\}$*. List*

(a) $A \cup B = \{1, 2, 3, 4, 5\}$

(b) $A \cap B = \{1, 3, 5\}$

(c) $A \setminus B = \{2, 4\}$

(d) $B \setminus A = \emptyset$

(e) $(A \times \{1\}) \cup (B \times \{2\}) = \{(1, 1), (2, 1), (3, 1), (4, 1), (5, 1), (1, 2), (3, 2), (5, 2)\}$

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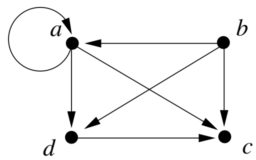
3. (8 points) Recall (Definition 2.14) that $G \subseteq A \times A$ is a *rooted graph* iff there is a node $r \in A$ such that for every $x \in A$ there is a path from r to x in G. In the graph to the right, which nodes can serve as r in this definition?



Only b can serve as the root of this graph. There is no path to b from any node—b has in-degree 0—neither a nor c nor d can be a root.

4. (10 points) Let $X = \{\texttt{a}, \texttt{b}\}$ and $Y = \{\texttt{ab}\}$ be alphabets. Note that X has two elements and Y has one element. Define the language $L \subseteq (X \cup Y)^+$ to be

$$L = \{u\hat{~}v\hat{~}u \mid u \in X^*,\ v \in Y^+\}$$

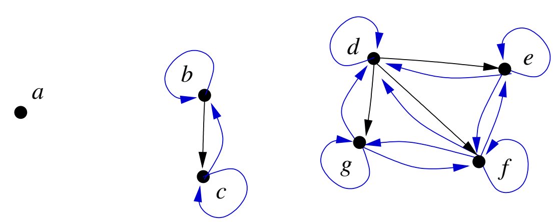
List all the words in L that are less than seven letters long.

{ab,
a^ab^a, b^ab^b,
a^a^ab^a^a, a^b^ab^a^b, b^a^ab^b^a, b^b^ab^b^b,
a^ab^ab^a, b^ab^ab^b,
a^a^ab^ab^a^a, a^b^ab^ab^a^b, b^a^ab^ab^b^a, b^b^ab^ab^b^b,
a^ab^ab^ab^a, b^ab^ab^ab^b}

5. (15 points) Recall that a relation $R \subseteq A \times A$ is:

reflexive iff $\forall x \in A\colon (x, x) \in R$.
symmetric iff $\forall (x, y) \in R\colon (y, x) \in R$.
transitive iff $\forall (x, y), (y, z) \in R\colon (x, z) \in R$.

(a) Add the minimal number of edges needed to make the graph $R \subset A \times A$, below, symmetric and transitive.



(b) Is the new relation reflexive? Is it irreflexive?

It is not reflexive because it does not contain (a, a). It is not irreflexive because, for one instance, it contains (b, b).

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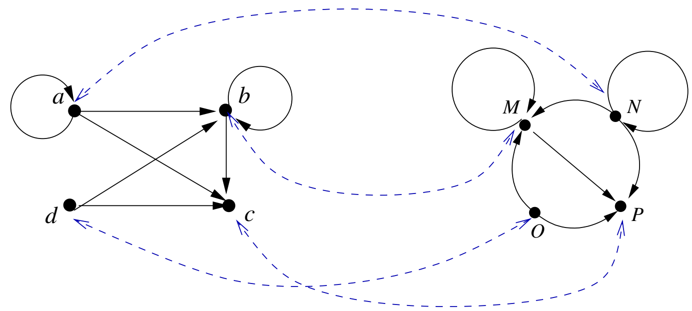
6. (15 points) Two graphs are *isomorphic* if they "have the same structure."

(a) Are the two graphs below isomorphic?



The function $f = \{(a,N),(b,M),(c,P),(d,O)\}$ is an (in fact, the only) isomorphism between the two relations. The picture above suffices as a solution, but in more detail,

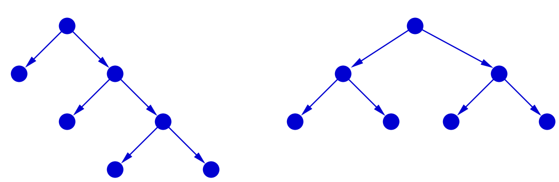
$f\colon A$	$\longrightarrow$	B
a	$\longmapsto$	N
b	$\longmapsto$	M
c	$\longmapsto$	P
d	$\longmapsto$	O

(x,y)	$\overset{?}{\longmapsto}$	$(f(x),f(y))$	
(a,a)	$\longmapsto$	(N,N)	$\surd$
(a,b)	$\longmapsto$	(N,M)	$\surd$
(a,c)	$\longmapsto$	(N,P)	$\surd$
(b,b)	$\longmapsto$	(M,M)	$\surd$
(b,c)	$\longmapsto$	(M,P)	$\surd$
(d,b)	$\longmapsto$	(O,M)	$\surd$
(d,c)	$\longmapsto$	(O,P)	$\surd$

(b) Write a formal definition of *isomorphism*.

Definition. *Two directed graphs* $R \subseteq A \times A$ *and* $S \subseteq B \times B$ *are* isomorphic *iff there exists a bijection* $f\colon A \to B$ *such that* $(x,x') \in R$ *iff* $(f(x),f(x')) \in S$*. The* f *is called an* isomorphism between R and S.

7. (10 points) A *binary tree* is a tree in which the out-degree of every node is either 0 or 2. Draw all the distinct (non-isomorphic) binary trees containing seven nodes.



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8. (10 points) A partial description of the STMT Programming Language is shown below. Write a program in the STMT programming language divides A by B using only addition and subtraction, leaving the quotient in program variable q and the remainder in r. It matters only that the program is correct; don't worry about efficiency. Your solution should satisfying the assertions

$\{x = A \in \mathbb{W} \text{ and } y = B \in \mathbb{W}\}$

```
begin
q := 0 ;
r := x ;
while r ≥ y do
  begin
  q := q + 1 ;
  r := r − y
  end
end
```

$\{qy + r = A \text{ and } r < B\}$

STMT language:

⟨STMT⟩	::=	⟨IDENTIFIER⟩ `:=` ⟨TERM⟩	(*assignment*)
	|	`if` ⟨TEST⟩ `then` ⟨STMT⟩ `else` ⟨STMT⟩	(*conditional*)
	|	`while` ⟨TEST⟩ `do` ⟨STMT⟩	(*repetition*)
	|	`begin` ⟨STMT⟩ `;` ... `;` ⟨STMT⟩ `end`	(*compound*)

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9. (10 points)

- Definition 2.5 states that the *composition* of two relations, $R \subseteq X \times Y$ and $S \subseteq Y \times X$ is

$$S \circ R = \{(x, z) \mid \exists y \in Y : (x, y) \in R \text{ and } (y, z) \in S\}$$

- Proposition 2.1 states that the composition of two functions is a function.
- Definition 2.6(b) says a function $f : X \to Y$ is *injective* iff

$$\forall x, x' \in X : f(x) = f(x') \text{ implies } x = x'$$

Prove: *The composition of two injections is an injection.*

COMMENT. *In the case of functions, we may write $f(x) = y$ instead of $(x, y) \in f$. Doing so makes the proof simpler to write and easier to read because the term $f(x)$ denotes the* unique "value of f at x."

PROOF. Suppose $f : X \to Y$ and $g : Y \to Z$ are injections. Suppose further that $(g \circ f)(x) = (g \circ f)(x')$. This means $g(f(x)) = g(f(x'))$. Since g is an injection it must be that $f(x) = f(x')$; and since f is an injection $f(x) = f(x')$ implies $x = x'$. We have shown that $(g \circ f)(x) = (g \circ f)(x')$ implies $x = x'$ satisfying Definition 2.6(b). Therefore, $g \circ f$ is an injection.

□