501 Midterm 2  
Nov 7, 2022

Start each problem on a new page.  
You may use major theorems from class without proof.  
In all problems $\Sigma = \{a, b, c\}$.

1. (20+10%)  
(a) Construct a Turing transducer $M$ over $\Sigma$ that doubles the first letter of the input (and leaves input $\epsilon$ unchanged). For example, for input $ba$ the output is $bbb$, and for input $bba$ the output is $bba$.  
You may give a modular presentation of $M$.  
\textbf{Solution.}  

\begin{align*}
S & \xrightarrow{\sigma(+) \sigma} S \\
S & \xrightarrow{\sigma(+) \tau} I_{\sigma} \quad (\sigma \in \Sigma) \\
I_{\sigma} & \xrightarrow{\tau(\sigma)} N_{\tau} \quad (\sigma, \tau \in \Sigma) \\
N_{\tau} & \xrightarrow{\sigma(+) \sigma} I_{\tau} \quad (\sigma, \tau \in \Sigma) \\
I_{\sigma} & \xrightarrow{\mu(\sigma)} P \quad (\sigma \in \Sigma)
\end{align*}

(b) Give the computation-trace of $M$ for input $acb$.  
\textbf{Solution.}  

\begin{align*}
(S, >acb) & \Rightarrow (I_c, >aab) \\
\Rightarrow (S, >acb) & \Rightarrow (N_b, >aac) \\
\Rightarrow (N_a, >acb) & \Rightarrow (I_b, >aac\mu) \\
\Rightarrow (I_a, >acb) & \Rightarrow (P, >aacb) \\
\Rightarrow (N_c, >aab)
\end{align*}

2. (10+10+10)  
(a) Construct a PDA $M$ recognizing the language $L = \{a^i b^j \mid i < j\}$. You may use a stack-bottom marker, but no other auxiliary symbol.  
\textbf{Solution.} Initial state $s$ and accept state $f$.  

\begin{align*}
s & \xrightarrow{\epsilon(\rightarrow s)} q \\
q & \xrightarrow{a(\rightarrow a)} q \\
q & \xrightarrow{b(\rightarrow s)} p \quad \text{one } b \text{ is freely amortized from input} \\
p & \xrightarrow{b(\rightarrow e)} p \\
p & \xrightarrow{\epsilon(\rightarrow s)} f \quad \text{extra } b \text{'s may be freely amortized}
\end{align*}

(b) Give a CFG $G$ generating $L$.  
\textbf{Solution.} Initial non-terminal $S$.  
\begin{align*}
S & \rightarrow aSb \mid Sb \mid b.
\end{align*}
(c) Using the algorithm for converting CFGs to equivalent PDAs obtain from $G$ another PDA $N$ recognizing $L$.

**Solution.** Initial state $s$ and accept state $f$.

$$
\begin{align*}
&s \xrightarrow{\epsilon \rightarrow S} c \\
&q \xrightarrow{a(a \rightarrow \epsilon)} q \\
&q \xrightarrow{b(b \rightarrow \epsilon)} q \\
&q \xrightarrow{\epsilon (S \rightarrow aSb)} q \\
&q \xrightarrow{\epsilon (S \rightarrow Sb)} q \\
&q \xrightarrow{\epsilon (S \rightarrow b)} q \\
&q \xrightarrow{\epsilon (\epsilon \rightarrow \epsilon)} a
\end{align*}
$$

3. (15%) For $w \in \Sigma^*$ let $w^{-\epsilon}$ be the result of eliminating in $w$ all occurrences of $c$; for example $w = abccac$ then $w^{-\epsilon} = aba$. For $L \subseteq \Sigma^*$ let $L^{-\epsilon} = \{ w^{-\epsilon} \mid w \in L \}$.

Show that if $L$ is a CFL then so is $L^{-\epsilon}$. You may either convert a CFG $G$ generating $L$ into a CFG $G'$ generating $L^{-\epsilon}$, or convert a PDA $M$ recognizing $L$ into a PDA $M'$ recognizing $L^{-\epsilon}$. You need not prove that your conversion works.

**Solution.** Using CFGs: Given a CFG $G$ generating $L$ let $G'$ be obtained from $G$ by replacing in productions $c$ by $\epsilon$. Then if $D$ is a derivation of $G$ of a string $X$ (allowing both terminals and non-terminals) then $D^{-\epsilon}$ is a derivation of $G^{-\epsilon}$ of $X^{-\epsilon}$. In particular $L(G') = L^{-\epsilon}$. (Equivalently: Take $c$ to be a non-terminal, and add to $G$ the production $c \rightarrow \epsilon$.)

4. (25%) Show that the following language is not CF.

$L = \{ a^i b^j c^i \mid i < j \}$.

**Solution.** We show that $L$ fails the Dual-Clipping Property.

Given $k > 0$, let $w = a^k b^{k+1} c^k$. Then $w \in L$ and $|w| \geq k$. If $p$ is any substring of $w$ of length $\leq k$ then $p$ cannot have both $a$'s and $c$'s.

Let $w'$ be the result of clipping off of $w$ some letters in $p$. If some $a$ is clipped then no $c$ is clipped, in which case $w'$ cannot have an equal number of $a$'s and $c$'s, thus $\notin L$. Similarly if some $c$ is clipped we obtain $w' \notin L$. Finally, if only $b$'s are clipped then $\#_a(w') = k \geq \#_a(w')$ and also $\notin L$.

Since $w' \notin L$ for any clipping within $p$ it follows that $L$ does not satisfy the Dual-Clipping Property, and is not CF.