B501 Practice for Midterm 1

Oct 16, 2024

For each of the following statements determine whether it is always true.

1. The inverse of a total mapping is surjective.

B501 Practice for Midterm 1

Oct 16, 2024

For each of the following statements determine whether it is always true.

1. The inverse of a total mapping is surjective.

   Solution. True. $F : A \Rightarrow B$ is total if every x in the domain of F, namely A, is mapped by F to some y in its range, i.e. B. This says precisely that every x in the range of F^{-1}, i.e. A, is mapped-to by F^{-1} from some y in B, i.e. its range. This says that F^{-1} is surjective.

B501 Practice for Midterm 1

Oct 16, 2024

For each of the following statements determine whether it is always true.

1. The inverse of a total mapping is surjective.
2. If there are injections $f\colon A \to B$ and $B \to A$ then $A \cong B$.

B501 Practice for Midterm 1

Oct 16, 2024

For each of the following statements determine whether it is always true.

1. The inverse of a total mapping is surjective.
2. If there are injections $f\colon A \to B$ and $B \to A$ then $A \cong B$.

Solution. Yes, That's the CBS Theorem

B501 Practice for Midterm 1

Oct 16, 2024

For each of the following statements determine whether it is always true.

1. The inverse of a total mapping is surjective.
2. If there are injections $f : A \to B$ and $B \to A$ then $A \cong B$.
3. Every DFA is an NFA.

B501 Practice for Midterm 1

Oct 16, 2024

For each of the following statements determine whether it is always true.

1. The inverse of a total mapping is surjective.
2. If there are injections $f : A \to B$ and $B \to A$ then $A \cong B$.
3. Every DFA is an NFA.

   Solution. True. A DFA is a special case of an NFA, where the transition mapping is a function.

B501 Practice for Midterm 1

Oct 16, 2024

For each of the following statements determine whether it is always true.

1. The inverse of a total mapping is surjective.
2. If there are injections $f: A \to B$ and $B \to A$ then $A \cong B$.
3. Every DFA is an NFA.
4. Every language denoted by a regular expression without star or plus is finite.

B501 Practice for Midterm 1

Oct 16, 2024

For each of the following statements determine whether it is always true.

1. The inverse of a total mapping is surjective.
2. If there are injections $f: A \to B$ and $B \to A$ then $A \cong B$.
3. Every DFA is an NFA.
4. Every language denoted by a regular expression without star or plus is finite.

   Solution. True. The initial regular expressions denote languages of size $\leqslant 1$, and union and concatenation, when applied to finite languages, yield finite languages.

B501 Practice for Midterm 1

Oct 16, 2024

For each of the following statements determine whether it is always true.

1. The inverse of a total mapping is surjective.
2. If there are injections $f: A \to B$ and $B \to A$ then $A \cong B$.
3. Every DFA is an NFA.
4. Every language denoted by a regular expression without star or plus is finite.
5. If L is a regular language then every sub-language of L is regular.

B501 Practice for Midterm 1

Oct 16, 2024

For each of the following statements determine whether it is always true.

1. The inverse of a total mapping is surjective.
2. If there are injections $f: A \to B$ and $B \to A$ then $A \cong B$.
3. Every DFA is an NFA.
4. Every language denoted by a regular expression without star or plus is finite.
5. If L is a regular language then every sub-language of L is regular.

   Solution. False. Every language over an alphabet Σ, including non-regular ones, is a subset of the regular languages Σ^*.

B501 Practice for Midterm 1

Oct 16, 2024

For each of the following statements determine whether it is always true.

1. The inverse of a total mapping is surjective.
2. If there are injections $f : A \to B$ and $B \to A$ then $A \cong B$.
3. Every DFA is an NFA.
4. Every language denoted by a regular expression without star or plus is finite.
5. If L is a regular language then every sub-language of L is regular.
6. If a set is infinite then it is countable.

B501 Practice for Midterm 1

Oct 16, 2024

For each of the following statements determine whether it is always true.

1. The inverse of a total mapping is surjective.
2. If there are injections $f : A \to B$ and $B \to A$ then $A \cong B$.
3. Every DFA is an NFA.
4. Every language denoted by a regular expression without star or plus is finite.
5. If L is a regular language then every sub-language of L is regular.
6. If a set is infinite then it is countable.

   Solution. False. For example, $\mathcal{P}(\mathbb{N})$ is not countable.

B501 Practice for Midterm 1

Oct 16, 2024

For each of the following statements determine whether it is always true.

1. The inverse of a total mapping is surjective.
2. If there are injections $f : A \to B$ and $B \to A$ then $A \cong B$.
3. Every DFA is an NFA.
4. Every language denoted by a regular expression without star or plus is finite.
5. If L is a regular language then every sub-language of L is regular.
6. If a set is infinite then it is countable.
7. If $F : A \rightharpoonup B$ then F^{-1} is injective.

B501 Practice for Midterm 1

Oct 16, 2024

For each of the following statements determine whether it is always true.

1. The inverse of a total mapping is surjective.
2. If there are injections $f: A \to B$ and $B \to A$ then $A \cong B$.
3. Every DFA is an NFA.
4. Every language denoted by a regular expression without star or plus is finite.
5. If L is a regular language then every sub-language of L is regular.
6. If a set is infinite then it is countable.
7. If $F: A \rightharpoonup B$ then F^{-1} is injective.

   Solution. True. If $F: A \rightharpoonup B$ then F is univalent (for every input x at most one output y) and therefore its inverse F^{-1} is injective (for every output x at most one input y.

B501 Practice for Midterm 1

Oct 16, 2024

For each of the following statements determine whether it is always true.

1. The inverse of a total mapping is surjective.
2. If there are injections $f : A \to B$ and $B \to A$ then $A \cong B$.
3. Every DFA is an NFA.
4. Every language denoted by a regular expression without star or plus is finite.
5. If L is a regular language then every sub-language of L is regular.
6. If a set is infinite then it is countable.
7. If $F : A \rightharpoonup B$ then F^{-1} is injective.
8. If L is recognized by an NFA, then its complement is basic.

B501 Practice for Midterm 1

Oct 16, 2024

For each of the following statements determine whether it is always true.

1. The inverse of a total mapping is surjective.
2. If there are injections $f : A \to B$ and $B \to A$ then $A \cong B$.
3. Every DFA is an NFA.
4. Every language denoted by a regular expression without star or plus is finite.
5. If L is a regular language then every sub-language of L is regular.
6. If a set is infinite then it is countable.
7. If $F : A \rightharpoonup B$ then F^{-1} is injective.
8. If L is recognized by an NFA, then its complement is basic.

   Solution. True. If L is recognized by an NFA then it is regular, so its complement is regular as well, and therefore basic.

B501 Practice for Midterm 1

Oct 16, 2024

For each of the following statements determine whether it is always true.

1. The inverse of a total mapping is surjective.
2. If there are injections $f : A \to B$ and $B \to A$ then $A \cong B$.
3. Every DFA is an NFA.
4. Every language denoted by a regular expression without star or plus is finite.
5. If L is a regular language then every sub-language of L is regular.
6. If a set is infinite then it is countable.
7. If $F : A \rightharpoonup B$ then F^{-1} is injective.
8. If L is recognized by an NFA, then its complement is basic.
9. The union of two non-regular languages is non-regular

B501 Practice for Midterm 1

Oct 16, 2024

For each of the following statements determine whether it is always true.

1. The inverse of a total mapping is surjective.
2. If there are injections $f: A \to B$ and $B \to A$ then $A \cong B$.
3. Every DFA is an NFA.
4. Every language denoted by a regular expression without star or plus is finite.
5. If L is a regular language then every sub-language of L is regular.
6. If a set is infinite then it is countable.
7. If $F: A \rightharpoonup B$ then F^{-1} is injective.
8. If L is recognized by an NFA, then its complement is basic.
9. The union of two non-regular languages is non-regular

   Solution. False. Take a non-regular $L \subset \Sigma^*$. Then $\bar{L}$ is non-regular, but the union is Σ^* which is regular.

1. Prove directly by induction on $\mathbb{N}$ that $\quad 2+4+\cdots+2n = n(n+1)$

2. Let $\Sigma = \{\mathsf{a}, \mathsf{b}\}$. Prove by induction on Σ^* that $(u \cdot v)^R = v^R \cdot u^R$ for all $u, v \in \Sigma^*$.

Solution. By induction on Σ^* we prove that for every $u \in \Sigma^*$ the following statement holds: for every $v \in \Sigma^*$ $(u \cdot v)^R = v^R \cdot u^R$.

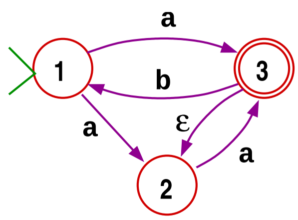
Base. $u = \varepsilon$. Then for every $v \in \Sigma^*$

$$
\begin{aligned}
(u \cdot v)^R &= (\varepsilon \cdot v)^R \\
&= v^R && \text{(dfn of concatenation)} \\
&= v^R \cdot \varepsilon \\
&= v^R \cdot \varepsilon^R && \text{dfn of reversal} \\
&= v^R \cdot u^R
\end{aligned}
$$

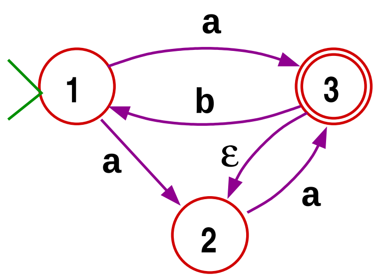
Step: Assume the identity for $u = w$. For $u = \sigma w$ $(\sigma \in \Sigma)$ we have then

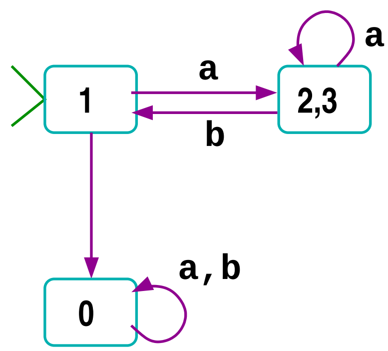
$$
\begin{aligned}
(u \cdot v)^R &= ((\sigma w) \cdot v)^R \\
&= (\sigma(w \cdot v))^R && \text{(dfn of concatenation)} \\
&= (w \cdot v)^R \cdot \sigma \\
&= (v^R \cdot w^R) \cdot \sigma && \text{(IH)} \\
&= v^R \cdot (w^R \cdot \sigma) \\
&= v^R \cdot (\sigma w)^R && \text{(dfn of reversal)} \\
&= v^R \cdot u^R
\end{aligned}
$$

3. Convert the following NFA to an equivalent DFA.



Solution.





4.

(a) Prove that the following language is regular:

$$L = \{x \cdot y \mid x, y \in \Sigma^*,\ |x| = |y|\} \quad .$$

Solution. L consists of the strings of even length, i.e. $L = (\Sigma \cdot \Sigma)^*$ which is regular since Σ is finite and the collection of regular languages is closed under concatenation and star.

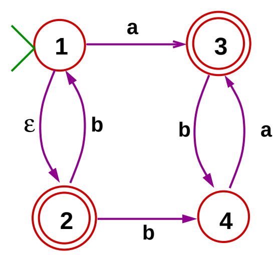
(b) Prove that the following language is not regular:

$$L = \{x\#y \mid x, y \in \Sigma^*,\ |x| = |y|\}$$

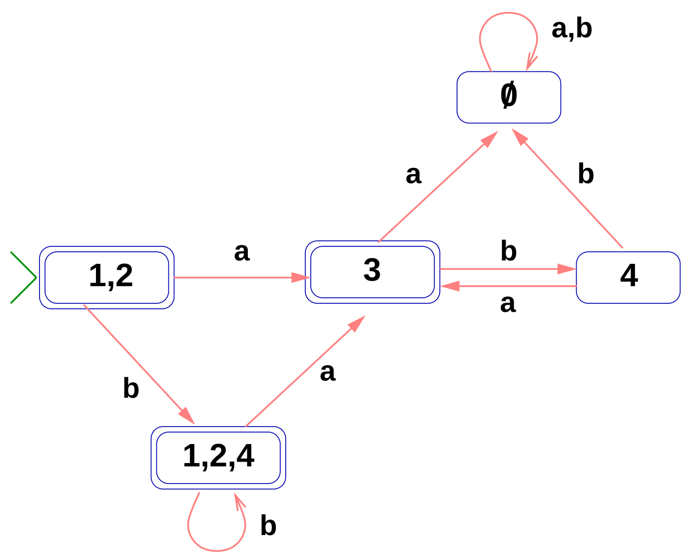
(where $\#$ is a fresh symbol).

Solution. This language is not regular since it fails the clipping property. Given k, let $w = \mathtt{a}^k\#\mathtt{b}^k$ and $u = \mathtt{a}^k$. Then $w \in L$ and u is a substring of length $\geqslant k$. But any clipping in u would yield a string $w' = \mathtt{a}^m\#\mathtt{b}^k$ with $m < k$, so $w' \notin L$.

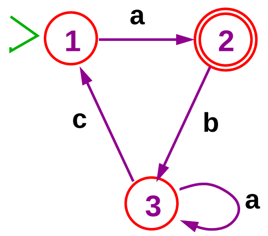
5. Convert the following NFA to an equivalent DFA. Label states of your DFA using the state-labels of the given NFA.



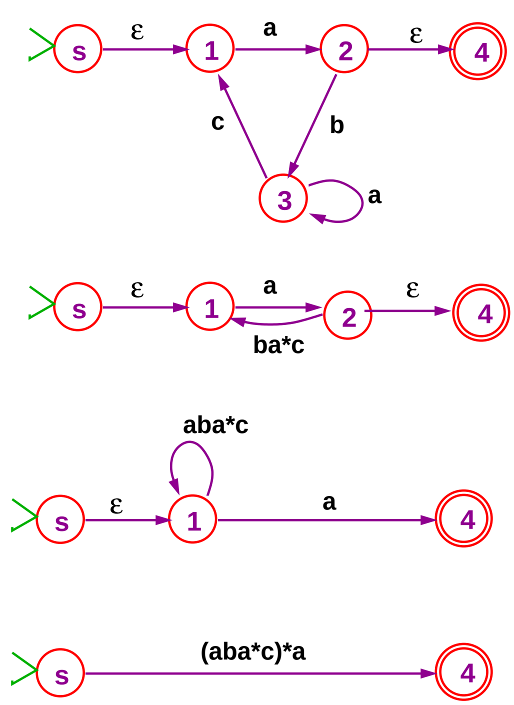
Solution.



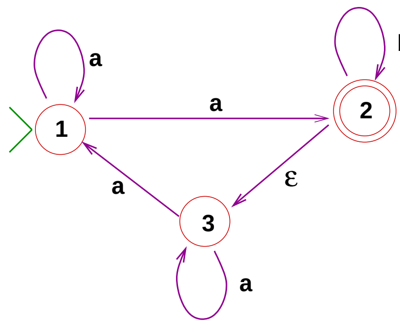
6. Use the state-elimination algorithm to obtain a regular expression that denotes the language below. Show all steps.



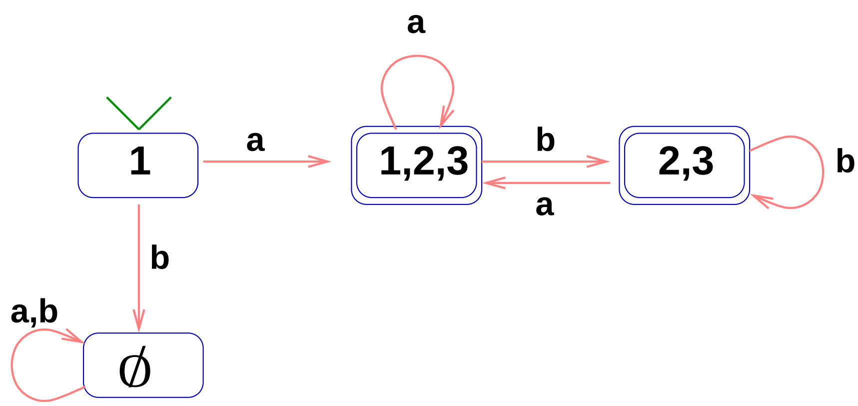
Solution.



7. Use the algorithm for NFA-to-DFA conversion to obtain a DFA equivalent to the following NFA.



Solution.



8. Let $L \subseteq \Sigma^*$ be regular. Show that the following languages are regular.

   (a) $\{\mathtt{a} \cdot x \mid x \in L\}$

   Solution. This language is $L \cdot \{\mathtt{a}\}$ which is regular as the concatenation of regular languages.

   (b) $\{x \mid \mathtt{a} \cdot x \in L\}$

   Solution. Since L is regular it is recognized by some DFA $M = (\Sigma, Q, s, A, \delta)$. Let $s \xrightarrow{a} q$ in M. Consider the DFA M' that differs from M only in having the state q above as its initial state. Then a string of the form $\mathtt{a}x$ is accepted by M, i.e. $s \xrightarrow{a} q \xrightarrow{x} A$ in M iff $q \xrightarrow{x} A$, i.e. iff M' accepts x. So M' recognizes $\{x \mid \mathtt{a} \cdot x \in L\}$, which is therefore regular.

   (c) $\{x \cdot \mathtt{a} \cdot y \mid x, y \in L\}$

   Solution. This language is $L \cdot \{\mathtt{a}\} \cdot L$ which is regular as the concatenation of regular languages.

9. For each of the following languages L over the alphabet $\Sigma = \{\mathtt{a}, \mathtt{b}\}$ identify its residues.

(a) $L = \mathcal{L}(\mathtt{a} \cdot \mathtt{b} \cdot \mathtt{a}^*)$

Solution.

$$
\begin{aligned}
L/\varepsilon &= L \\
L/\mathtt{a} &= \mathcal{L}(\mathtt{ba}^*) \\
L/\mathtt{ab}u &= \mathcal{L}(\mathtt{a}^*) && \text{for all } u \in \mathcal{L}(\mathtt{a}^*) \\
L/w &= \emptyset && \text{for any other } w
\end{aligned}
$$

(b) $L = \{\mathtt{ab}, \mathtt{ba}, \mathtt{aaba}\}$

Solution.

$$
\begin{aligned}
L/\varepsilon &= L \\
L/\mathtt{a} &= \{\mathtt{b}, \mathtt{aba}\} \\
L/\mathtt{b} &= L/\mathtt{aab} = \{\mathtt{a}\} \\
L/\mathtt{ab} &= L/\mathtt{ba} = L/\mathtt{aaba} = \{\varepsilon\} \\
L/\mathtt{aa} &= \{\mathtt{ba}\} \\
L/w &= \emptyset && \text{for any other } w
\end{aligned}
$$

(c) $L = \{w \in \{\mathtt{a}, \mathtt{b}\}^* \mid \#_a(w) \text{ is even, } \#_b(w) \text{ odd } \}$