B501, Fall 2024

501 Midterm

Oct 21, 2024

Start each problem on a new page. We write here Σ for the alphabet $\{\mathtt{a},\mathtt{b}\}$.

1. (25%) For each of the following statements determine whether it is always true. (***No credit*** without adequate explanation.)

   (a) If $F : A \Rightarrow B$ is injective then so is $F^{-1} : B \Rightarrow A$.

   Solution. False. The mapping $F : \{1\} \to \{0,1\}$ that maps 1 to both 0 and 1 is injective, but its inverse maps both 0 and 1 to 0 and so is not injective.

   (b) For every set S there is an injection $j : S \to \mathbb{R}$.

   Solution. False. If there were an injection from $\mathcal{P}(\mathbb{R})$ to $\mathbb{R}$ then we'd have $\mathcal{P}(\mathbb{R}) \cong \mathbb{R}$ by CBS, because the mapping $a \mapsto \{a\}$ is an injection from $\mathbb{R}$ to $\mathcal{P}(\mathbb{R})$.

   (c) If L and K are languages denoted by regular expressions then so is their intersection.

   Solution. True. If L, K are denoted by regular expressions then they are regular, and so their intersection is regular. And every regular language is denoted by some regular expression.

   (d) Every language denoted by a regular expression without star is finite.

   Solution. True. The initial regular expressions denote languages of size $\leqslant 1$, and union and concatenation, when applied to finite languages, yield finite languages. (The midterm's phrasing of this problem was ambiguous about using the plus operation (+). Credit was given for negative answers based on this.)

   (e) If every string in L is finite then L is recognized by an NFA.

   Solution. False. Every string is finite, so the premise is vacuously true of any language L, regular or not.

2. (15%) Prove that $[0..1] \cong (0..1)$, i.e. the closed interval $[0..1]$ is equipollent to the open interval $(0..1)$.

   Solution. $(0..1) \preccurlyeq [0..1]$ since the identity mapping from $(0..1)$ to $[0..1]$ is an injection.

   On the other hand, $[0..1] \preccurlyeq (-1,2)$ by the identity injection, and $(-1,2) \preccurlyeq (0..1)$ by the injection $x \mapsto (x+1)/3$. Composing the two we get $[0..1] \preccurlyeq (0..1)$.

   By the CBS Theorem it follows that $[0..1] \cong (0..1)$.

3. (3+3+9%)

(a) Give a generative definition of the set D of positive odd integers.

Solution. $1 \in D$ and if $x \in D$ then $x+1+1 \in D$.

(b) Referring to your definition, state an induction principle for D .

Solution. Assume that a property P of positive odd integers is true of 1 , and whenever it is true of x then it is also true of $x+2$. Conclude that P is true of all elements of D.

(c) Using your induction principle show that $n^2 - 1$ is divisible by 8 for every odd integer. (Note: If n is odd then $n{+}1$ is even!)

Solution. ***Basis.*** For $n = 1$ we have $n^2 - 1 = 0$ which is divisible by 8.
Step. Assume (IH) that for $n = k$ we have $n^2 - 1$, i.e. $k^2 - 1$, is divisible by 8. Then for $n = k{+}2$ we have

$$
\begin{array}{rcl}
n^2 - 1 & = & (k+2)^2 - 1 \\
 & = & k^2 + 4k + 4 - 1 \\
 & = & (k^2 + 1) + 4(k+1)
\end{array}
$$

The first addend is divisible by 8 by IH, and the second is divisible by 8 since $k{+}1$ is even. So $n^2 - 1$ is divisible by 8 for $n = k{+}2$.
By induction on D it follows that $n^2 - 1$ is divisible by 8 for all $n \in D$.

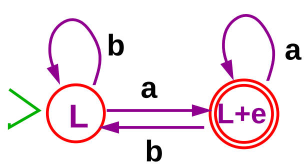
4. (15%) Let $\Sigma = \{\mathtt{a}, \mathtt{b}\}$ and $L = \Sigma^* \cdot \{\mathtt{a}\}$.

(a) Identify the residues of L.

Solution. $L/w = L$ for every w not ending with $\mathtt{a}$.
$L/w = L \cup \{\varepsilon\}$ for every w ending with $\mathtt{a}$.

(b) Construct a DFA recognizing L whose states are the residues.

Solution.



5. (15%) Prove that the following language is not regular. $L = \{\mathtt{a}^p \cdot \mathtt{b}^q \mid p < q\}$.

Solution. L fails the Clipping Property. Given $k > 0$ let $w = \mathtt{a}^k\mathtt{b}^{k+1}$ and let u be w's substring $\mathtt{b}^k + 1$. Then $w \in L$ and $|u| \geqslant k$. Any non-empty clipping in u yields from w a string $w' = \mathtt{a}^k\mathtt{b}^m$ with $m \leqslant k$, which is not in L .

Since L fails the Clipping Property, it is not regular.

6. (15%) Use the state-elimination algorithm to obtain a regular expression that denotes the language recognized by the following NFA. Show all steps.