MATHEMATICAL MACHINES

Computing

- Most computing consists in actions that modify data:
  - The data is textual
  - The actions are discrete: well-defined and single-step.

Computing

- Most computing consists in actions that modify data:
  - The data is textual
  - The actions are discrete: well-defined and single-step.
- The data is textual because discrete data has textual representation. (Though not all computing is discrete, eg Analog Computing is not.)

Acceptors

- *What algorithms do.*

Acceptors

- *What algorithms do.*
- Two main options: acceptors and transducers.
- An ***acceptor*** is an algorithm that takes a textual input
  (representing input data)
  and upon termination may or may not issue ***accept*** as output.

Acceptors

- *What algorithms do.*
- Two main options: acceptors and transducers.
- An ***acceptor*** is an algorithm that takes a textual input
  (representing input data)
  and upon termination may or may not issue ***accept*** as output.
- An acceptor that terminates for all input is a ***decider.***
- When a decider terminate for an input without accepting
  we say that it ***rejects*** the input.
- A decider is thus a solution for a decision problem.

Transducers

- A ***transducer*** is an algorithm that takes strings as input, and upon termination yields a string as output.

Transducers

- A ***transducer*** is an algorithm that takes strings as input, and upon termination yields a string as output.
- A transducer computes a ***partial-function*** (i.e. univalent mapping).

Transducers

- A ***transducer*** is an algorithm that takes strings as input, and upon termination yields a string as output.
- A transducer computes a ***partial-function*** (i.e. univalent mapping).
- An acceptor can be viewed as a transducer with ***accept*** as the only possible output;

  and a decider as a total transducer with ***accept*** and ***reject*** as the only possible outputs.

The simplest devices

- *What is the simplest possible mathematical machine:*
  - Transducer, or acceptor?

The simplest devices

- *What is the simplest possible mathematical machine:*
  - Transducer, or acceptor?
  - Fixed, or expandable external memory?

The simplest devices

- *What is the simplest possible mathematical machine:*
  - Transducer, or acceptor?
  - Fixed, or expandable external memory?
  - Random-access, or sequential reading?

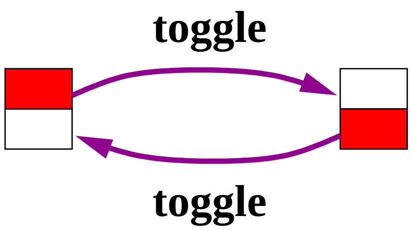
The simplest devices

- *What is the simplest possible mathematical machine:*
  - Transducer, or acceptor?
  - Fixed, or expandable external memory?
  - Random-access, or sequential reading?
- We start with the ***automaton,***
  an acceptor with no external memory that reads its input sequentially!

The simplest devices

- *What is the simplest possible mathematical machine:*
  - Transducer, or acceptor?
  - Fixed, or expandable external memory?
  - Random-access, or sequential reading?
- We start with the ***automaton,***
  an acceptor with no external memory that reads its input sequentially!
- This model captures the behavior of
  many familiar physical devices.
  Let's look at a couple of very simple ones.

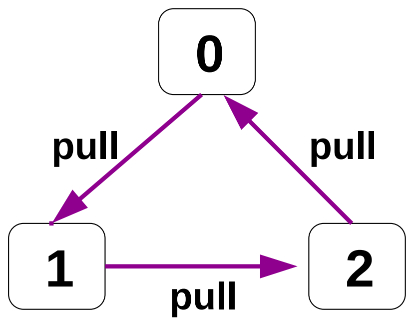
The electric switch



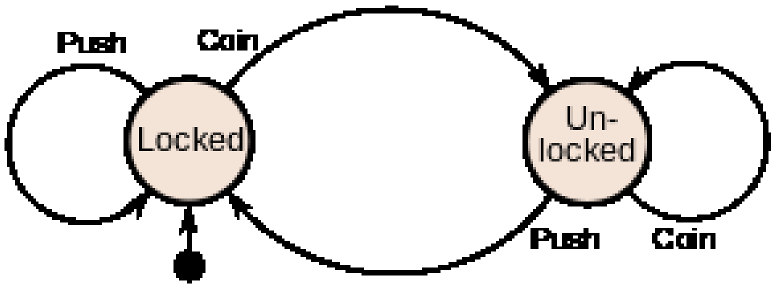
- The position of the switch is inverted
  after an odd number of toggles,
  and remains unchanged after an even number.

The ceiling fan

- A ceiling fan with manual cord-controlled:
  The speed is incremented (mod 2) with each pull.



The toll-turnstile



- The turnstile can be in one of two states: locked or unlocked.
- The action *insert token*
  changes the state *locked* into *unlocked*.
- The action *push and pass*
  changes the state *unlocked* into *locked*.

States

- A core concept of mathematical machines is the ***state.***
- E.g. a state of an elevator might consist of
  its position, motion (up, down, rest), upcoming destinations, time idle, etc.
- States are often labeled, for convenience, but don't have to be.

States

- A core concept of mathematical machines is the ***state.***
- E.g. a state of an elevator might consist of
  its position, motion (up, down, rest), upcoming destinations, time idle, etc.
- States are often labeled, for convenience, but don't have to be.
- Given a practical problem, deciding what are the relevant "states"
  often requires careful analysis.
- But once a mathematical model is distilled,
  the ***states*** become an abstraction,
  which we can represent graphically, e.g. by a circle.

Transitions

- A ***transition-rule*** is a mapping from states to states. We label each transition-rule by an identifier.

Transitions

- A ***transition-rule*** is a mapping from states to states. We label each transition-rule by an identifier.
- We focus for now on transitions that are ***functions***, i.e. univalent and total.

Transitions

- A **transition-rule** is a mapping from states to states. We label each transition-rule by an identifier.
- We focus for now on transitions that are ***functions***, i.e. univalent and total.
- A pair of states related by a transition-rule `a` is an ***action*** of `a`.

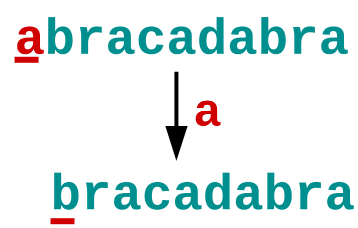
Transitions

- A ***transition-rule*** is a mapping from states to states. We label each transition-rule by an identifier.
- We focus for now on transitions that are ***functions***, i.e. univalent and total.
- A pair of states related by a transition-rule `a` is an ***action*** of `a`.
- For the toll-turnstile and the stopwatch the transition-rules are determined by certain human actions.

Textual form of transitions

- Since all finite discrete structures have simple textual codes, we can assume that:

  1. All input data is textual
  2. Each transition is coded by a single reserved letter
  3. The action of the transition labeled `a` is the reading (i.e. consumption) of `a`, much like the movement of a cursor.



A transition system

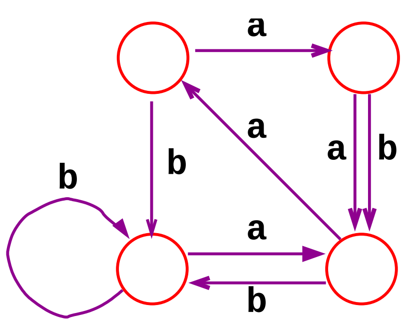
- A ***transition-system*** consists of a set of states and transition-rules over them.

A transition system

- A ***transition-system*** consists of a set of states
  and transition-rules over them.
- So a transition-system can be represented as a labeled di-graph:
  The nodes are the states,
  and the the actions are labeled edges.

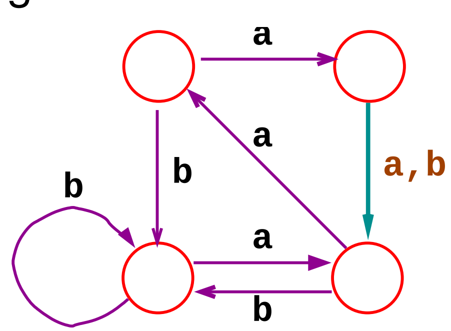
A transition system

- A ***transition-system*** consists of a set of states and transition-rules over them.
- So a transition-system can be represented as a labeled di-graph: The nodes are the states, and the the actions are labeled edges.
- When all transition-rules are functions, there is exactly one edge for each state and action:



A transition system

- A ***transition-system*** consists of a set of states
  and transition-rules over them.
- So a transition-system can be represented as a labeled di-graph:
  The nodes are the states,
  and the the actions are labeled edges.
- When all transition-rules are functions,
  there is exactly one edge for each state and action:



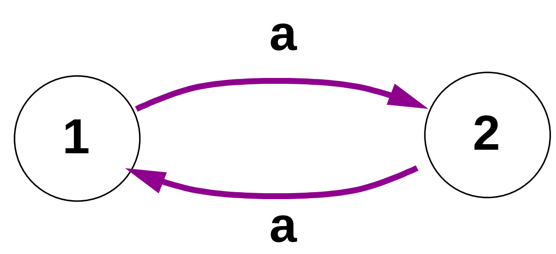
We merge arrow-labels for readability.

Example: Detecting an odd number of actions

- Consider the switch.

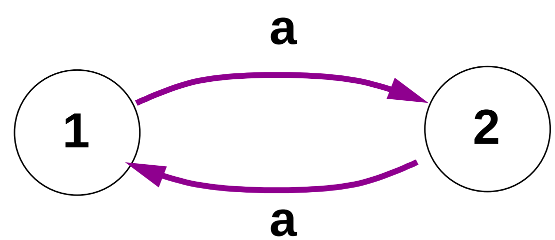
  We represent the transition “toggle” by the letter `a` ,

  and label the states as 1 and 2:



Example: Detecting an odd number of actions

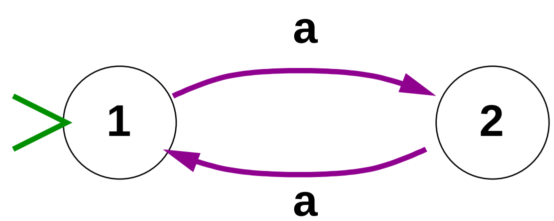
- Consider the switch.
  We represent the transition “toggle” by the letter `a` ,
  and label the states as 1 and 2:



- The device reads strings of `a`’s,
  and with each letter read it switch state.
- Reading odd number of `a` ’s leads to the opposite state.
- The physical nature of the toggle action is no longer present,
  and is indeed irrelevant.

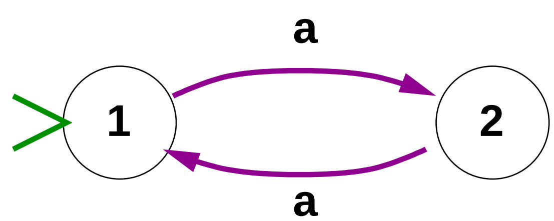
Start state and accepting states

- We intend to start at a particular state,
  so we single out one state as the ***initial*** (starting) state,
  indicated graphically by an incoming arrow.



Start state and accepting states

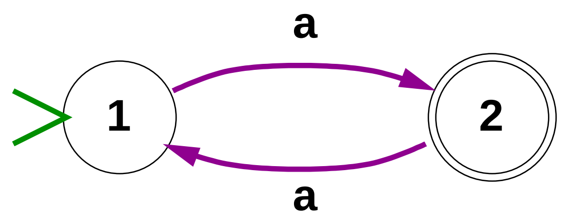
- We intend to start at a particular state,
  so we single out one state as the ***initial*** (starting) state,
  indicated graphically by an incoming arrow.



Where do the strings of length 1,3,... odd n lead?

Start state and accepting states

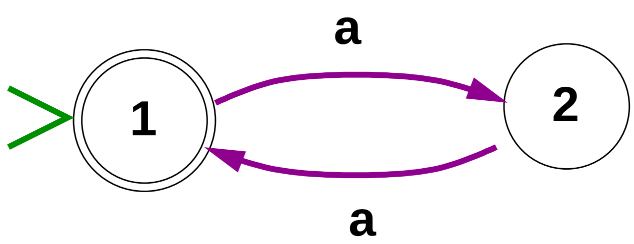
- We intend to start at a particular state,
  so we single out one state as the ***initial*** (starting) state,
  indicated graphically by an incoming arrow.



- The strings of odd length leads to state 2,
  so to accept just those strings we'd set 2
  as the unique accepting state.
- We do this graphically by doubling the contour of state 2.
- In general there can be several accepting states.

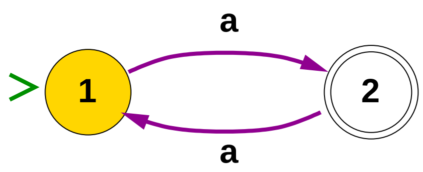
Initial state can be accepting

- It is possible that the initial state is accepting.
- To accept the strings of even length
  set **1** as the only accepting state:

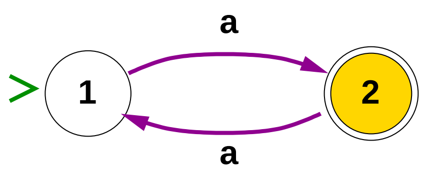


The device in action

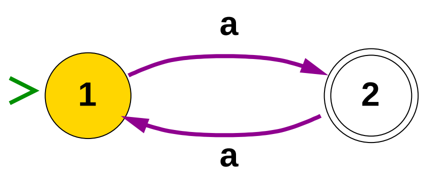
- Device accepting odd length:



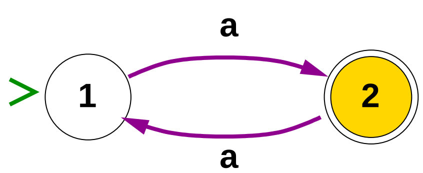
READING



a



aa

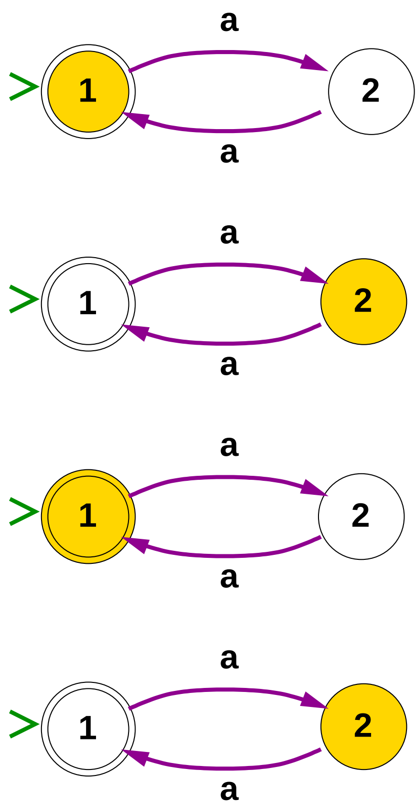


aaa

string accepted IFF has odd #a
aaa accepted

The device in action

- Device accepting even length:



READING

a

aa

aaa

string accepted IFF has even #a
aaa not accepted

Definition of automata

- An ***automaton,*** aka ***deterministic finite automaton (DFA)*** consists of
  - An alphabet Σ.

Definition of automata

- An ***automaton,*** aka ***deterministic finite automaton (DFA)*** consists of
  - An alphabet Σ.
  - A non-empty finite set Q of objects called ***states***.
  - One state $s \in Q$ singled out as ***initial-state*** (or ***initial-state***).
  - A set $A \subseteq S$ of states singled out as ***accepting states***.

Definition of automata

- An ***automaton,*** aka ***deterministic finite automaton (DFA)*** consists of

  - An alphabet Σ.
  - A non-empty finite set Q of objects called ***states***.
  - One state $s \in Q$ singled out as ***initial-state*** (or ***initial-state***).
  - A set $A \subseteq S$ of states singled out as ***accepting states***.
  - A ***transition function*** $\delta: Q \times \Sigma \to Q$.
    Given state $q \in Q$ and input-symbol σ
    $\delta(q, \sigma)$ is the new (target) state.

- We also write $q \xrightarrow{\sigma} p$ for $\delta(q, \sigma) = p$.
  Note: p may be the same as q.

Comments on the definition

- Formally, M above is a tuple $(\Sigma, Q, s, A, \delta)$ of its components.

Comments on the definition

- Formally, M above is a tuple $(\Sigma, Q, s, A, \delta)$ of its components.
- M is ***over the alphabet*** Σ.
  We don't mention Σ when irrelevant or clear.

Comments on the definition

- Formally, M above is a tuple $(\Sigma, Q, s, A, \delta)$ of its components.
- M is ***over the alphabet*** Σ.
  We don't mention Σ when irrelevant or clear.
- *Automaton* is of Greek origin:
  auto = self, *matos* = move.
  Plural: *automata* or *automatons*. *Automata* is never singular.

Comments on the definition

- Formally, M above is a tuple $(\Sigma, Q, s, A, \delta)$ of its components.
- M is ***over the alphabet*** Σ.
  We don't mention Σ when irrelevant or clear.
- *Automaton* is of Greek origin:
  auto = self, *matos* = move.
  Plural: *automata* or *automatons*. *Automat***a** is never singular.
- Since automata play a central role,
  they've acquired over time several alternative names, in particular *deterministic finite automaton (DFA)*.which we'll frequently use.

Some practical applications of automata

Textual applications

- Pattern matching, search engines
- Lexical analysis for compilation
- Data compression
- Automatic translation

Some practical applications of automata

Software systems

- Cyber-security
- System planning
- Information streaming
- Bio-informatics

Some practical applications of automata

Hardware systems

- Circuit design
- Robotics

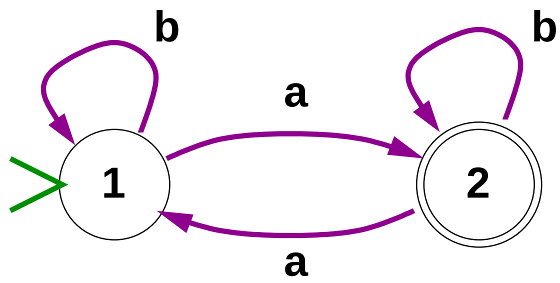
Some practical applications of automata

Verification

- System modeling
- Verification of communication protocols
- Verification of embedded systems
- Model checking

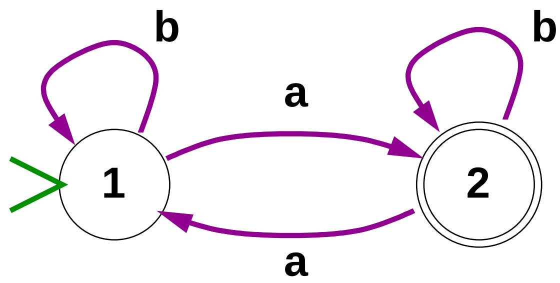
Example of a formal description

- Here's an automaton M over $\Sigma = \{\mathtt{a}, \mathtt{b}\}$ that accepts strings with an odd number of $\mathtt{a}$'s (and no others).



Example of a formal description

- Here's an automaton M over $\Sigma = \{\mathtt{a}, \mathtt{b}\}$ that accepts strings with an odd number of $\mathtt{a}$'s (and no others).



- Its formal definition: $M = (\Sigma, Q, s, A, \delta)$ where
  - ⋆ $\Sigma = \{\mathtt{a}, \mathtt{b}\}$
  - ⋆ $Q = \{1, 2\}$
  - ⋆ $s = 1$
  - ⋆ $A = \{2\}$

Operational semantics: How automata function

- Intuitively, an automaton reads successive input symbols
  starting with the initial state, and
  updating the state according to the transition function δ.

Operational semantics: How automata function

- Intuitively, an automaton reads successive input symbols
  starting with the initial state, and
  updating the state according to the transition function δ.
- The steps of an automaton change just the state,
  and the implicit move to the next input symbol.
- Since the transition mapping of an automaton is a function,
  there is exactly one next-state for each symbol read.

Operational semantics: How automata function

- Intuitively, an automaton reads successive input symbols
  starting with the initial state, and
  updating the state according to the transition function δ.
- The steps of an automaton change just the state,
  and the implicit move to the next input symbol.
- Since the transition mapping of an automaton is a function,
  there is exactly one next-state for each symbol read.
- Computation terminates iff the end of the input string is reached.

Operational semantics: How automata function

- Intuitively, an automaton reads successive input symbols
  starting with the initial state, and
  updating the state according to the transition function δ.
- The steps of an automaton change just the state,
  and the implicit move to the next input symbol.
- Since the transition mapping of an automaton is a function,
  there is exactly one next-state for each symbol read.
- Computation terminates iff the end of the input string is reached.
- The essence of a DFA is in its being an ***online acceptor***.

Traces

- If $w = \sigma_1 \cdots \sigma_n$ then we write $q \xrightarrow{\sigma_1 \cdots \sigma_n} p$
  to state that

$$q \xrightarrow{\sigma_1} r_1 \xrightarrow{\sigma_2} r_2 \quad \cdots \quad r_{n-1} \xrightarrow{\sigma_n} p$$

for some states $r_1, \ldots, r_{n-1}$.

Traces

- If $w = \sigma_1 \cdots \sigma_n$ then we write $q \xrightarrow{\sigma_1 \cdots \sigma_n} p$
  to state that
  $$q \xrightarrow{\sigma_1} r_1 \xrightarrow{\sigma_2} r_2 \quad \cdots \quad r_{n-1} \xrightarrow{\sigma_n} p$$
  for some states $r_1, \ldots, r_{n-1}$.

- The sequence of states $q, \; r_1, \; r_2, \; \cdots \; r_{n-1}, \; p$
  is a ***state-trace*** of the automaton.

Inductive definition of traces

- The ternary relation $q \xrightarrow{w} p$ can be defined inductively, by recurrence on w :
  - $q \xrightarrow{\varepsilon} q$
  - If $\delta(q, \sigma) = p$ that is $q \xrightarrow{\sigma u} r$, and $p \xrightarrow{u} r$ then $p \xrightarrow{\sigma} q$.

Inductive definition of traces

- The ternary relation $q \xrightarrow{w} p$ can be defined inductively, by recurrence on w:
  - $q \xrightarrow{\varepsilon} q$
  - If $\delta(q,\sigma) = p$ that is $q \xrightarrow{\sigma u} r$, and $p \xrightarrow{u} r$ then $p \xrightarrow{\sigma} q$.
- This definition invokes no auxiliary data that might be modified during execution.
- No mathematical machine we'll encounter (except NFAs) has such a definition:

  They all are based on a notion of ***configuration***, which combines the machine's states with modifiable data.

Accepted strings, recognized languages

- For $A \subseteq Q$ let's write $q \xrightarrow{w} A$
  when $q \xrightarrow{w} p$ for some $p \in A$.
- M **accepts** w when $s \xrightarrow{w} A$.

Accepted strings, recognized languages

- For $A \subseteq Q$ let's write $q \xrightarrow{w} A$ when $q \xrightarrow{w} p$ for some $p \in A$.
- M **accepts** w when $s \xrightarrow{w} A$.
- The language **recognized** by M is

$$\mathcal{L}(M) = \{w \in \Sigma^* \mid M \text{ accepts } w\} = \{w \in \Sigma^* \mid s \xrightarrow{w} A\}$$

- We re-use here the notation $\mathcal{L}(\cdots)$ that we used for regular expressions.

Accepted strings, recognized languages

- For $A \subseteq Q$ let's write $q \xrightarrow{w} A$
  when $q \xrightarrow{w} p$ for some $p \in A$.
- M **accepts** w when $s \xrightarrow{w} A$.
- The language **recognized** by M is

$$\mathcal{L}(M) = \{w \in \Sigma^* \mid M \text{ accepts } w\}$$
$$= \{w \in \Sigma^* \mid s \xrightarrow{w} A\}$$

- We re-use here the notation $\mathcal{L}(\cdots)$ that we used for regular expressions.
- Two automata are **equivalent** if they recognize the same language.

Automata are strictly regimented

1. Automata are acceptors: they produce no output.

Automata are strictly regimented

1. Automata are acceptors: they produce no output.
2. The input must be lexical (strings over a fixed alphabet).

Automata are strictly regimented

1. Automata are acceptors: they produce no output.
2. The input must be lexical (strings over a fixed alphabet).
3. Scanning forward: no backtracking or repositioning.

Automata are strictly regimented

1. Automata are acceptors: they produce no output.
2. The input must be lexical (strings over a fixed alphabet).
3. Scanning forward: no backtracking or repositioning.
4. Scanning at a single point (i.e. computation is ***on-line***).

Automata are strictly regimented

1. Automata are acceptors: they produce no output.
2. The input must be lexical (strings over a fixed alphabet).
3. Scanning forward: no backtracking or repositioning.
4. Scanning at a single point (i.e. computation is ***on-line***).
5. Exactly one move exists for each state and symbol.

Automata are strictly regimented

1. Automata are acceptors: they produce no output.
2. The input must be lexical (strings over a fixed alphabet).
3. Scanning forward: no backtracking or repositioning.
4. Scanning at a single point (i.e. computation is ***on-line***).
5. Exactly one move exists for each state and symbol.
6. Computation stops when the input's end is reached.

Automata are strictly regimented

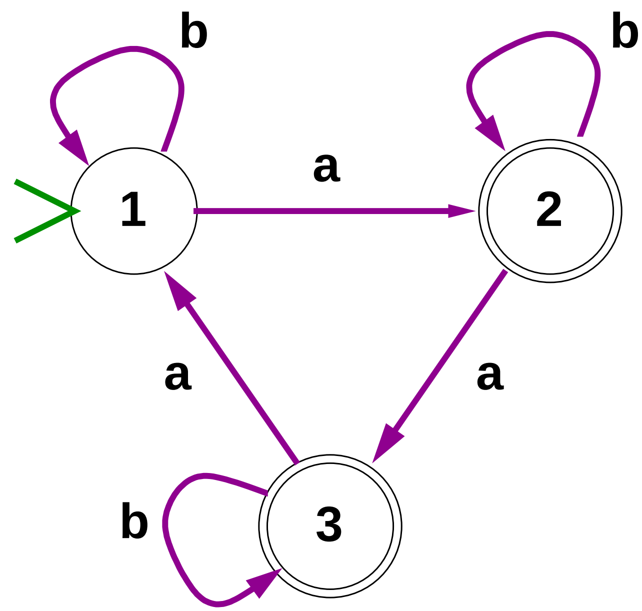
1. Automata are acceptors: they produce no output.
2. The input must be lexical (strings over a fixed alphabet).
3. Scanning forward: no backtracking or repositioning.
4. Scanning at a single point (i.e. computation is ***on-line***).
5. Exactly one move exists for each state and symbol.
6. Computation stops when the input's end is reached.
7. No auxiliary memory or devices.

Automata are strictly regimented

Only two are crucial: violating them changes computing's nature:

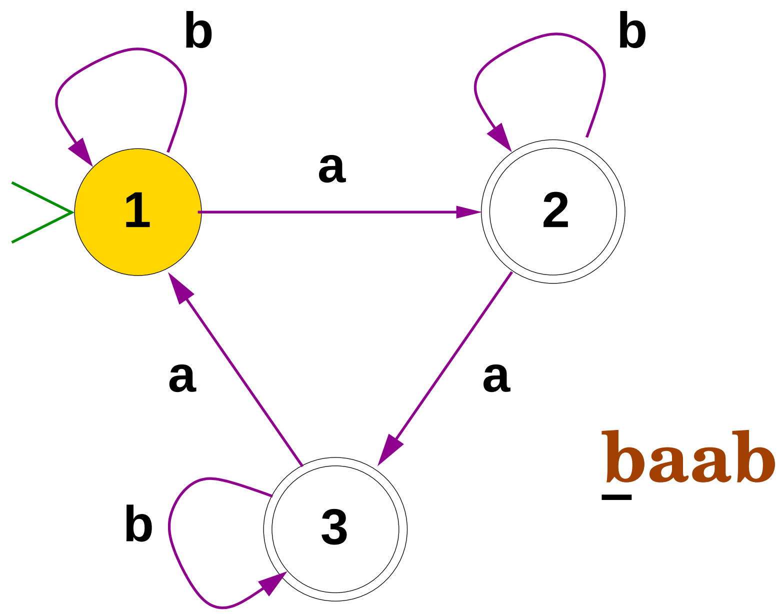
1. Automata are acceptors: they produce no output.
2. The input must be lexical (strings over a fixed alphabet).
3. Scanning forward: no backtracking or repositioning.
4. Scanning at a single point (i.e. computation is ***on-line***).
5. Exactly one move exists for each state and symbol.
6. Computation stops when the input's end is reached.
7. No auxiliary memory or devices.

Example: An automaton for Mod 3



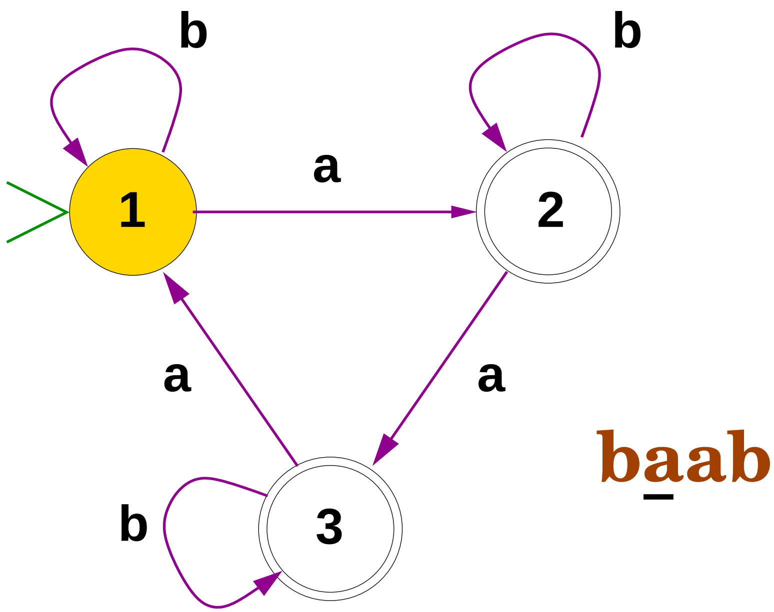
- $w \in \{\mathtt{a}, \mathtt{b}\}^*$ accepted iff $\#_a(w) \neq 0 \pmod 3$

Example of an accepted string



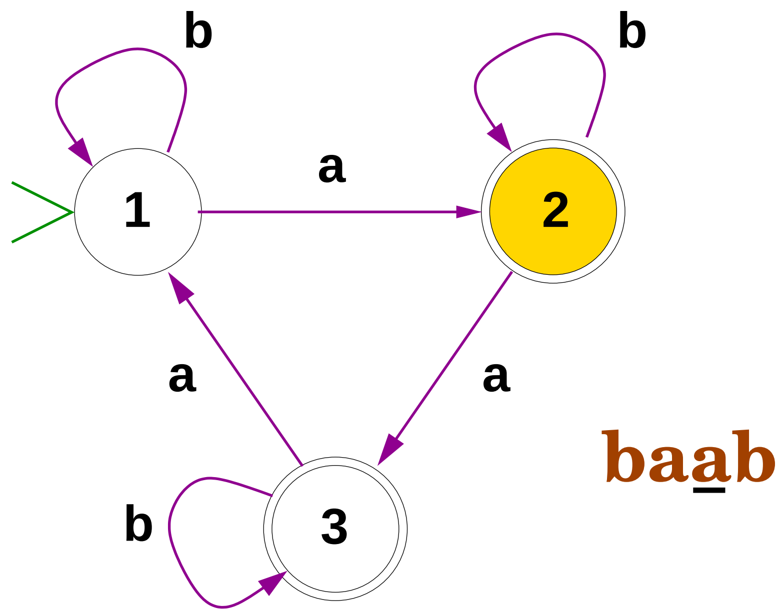
- State 1 (initial). Nothing read yet.

An accepted string



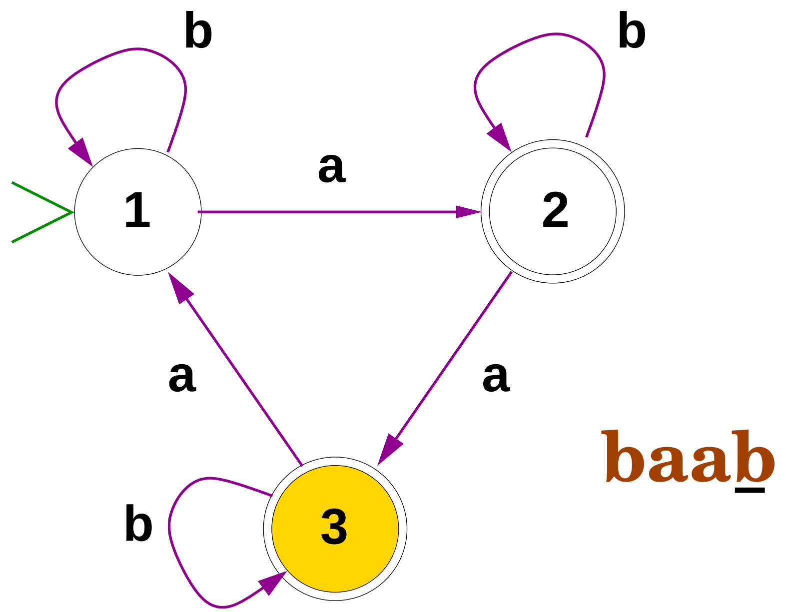
- Still state 1. Initial b read.

An accepted string



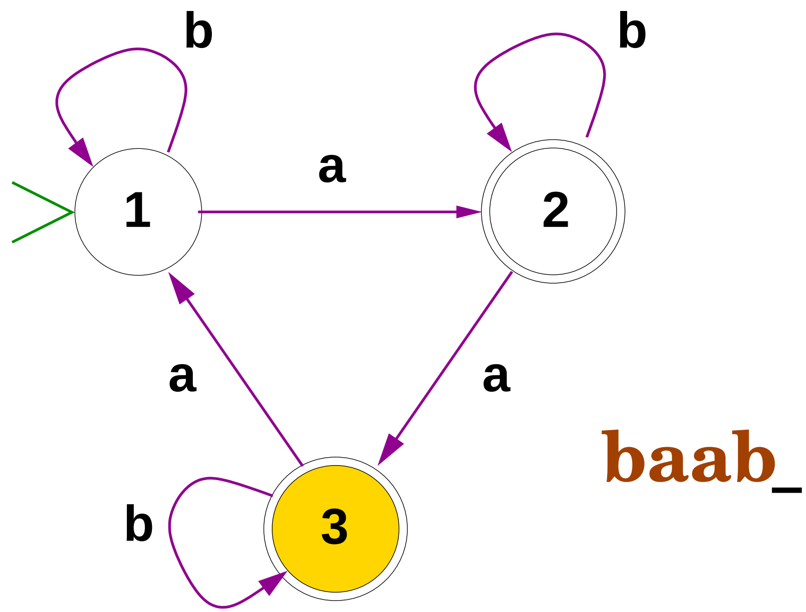
- Read `ba`, state 2.

An accepted string



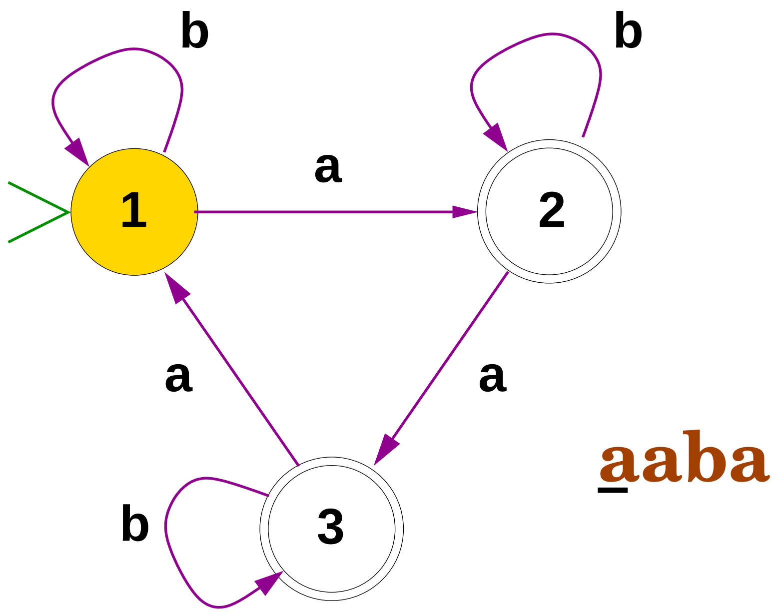
- Read baa, state 3.

An accepted string



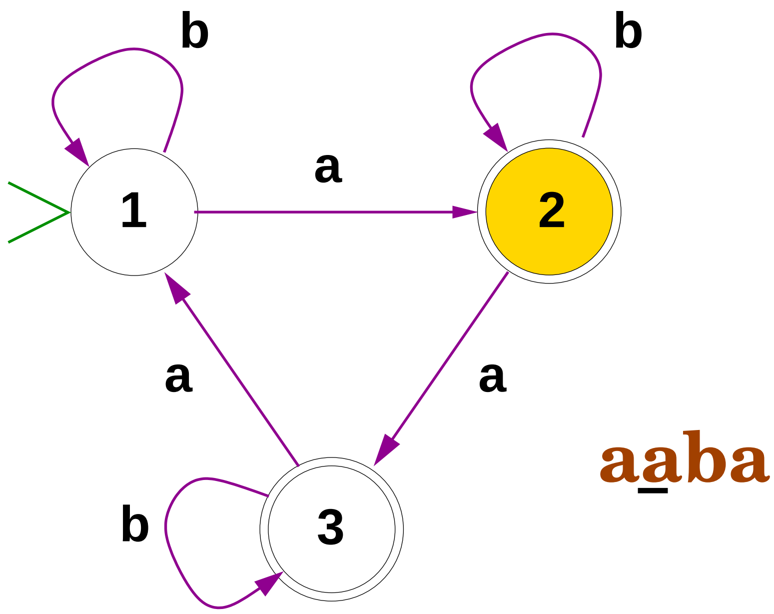
- Finished reading *baab*, state 3, accepted.

A non-accepted string



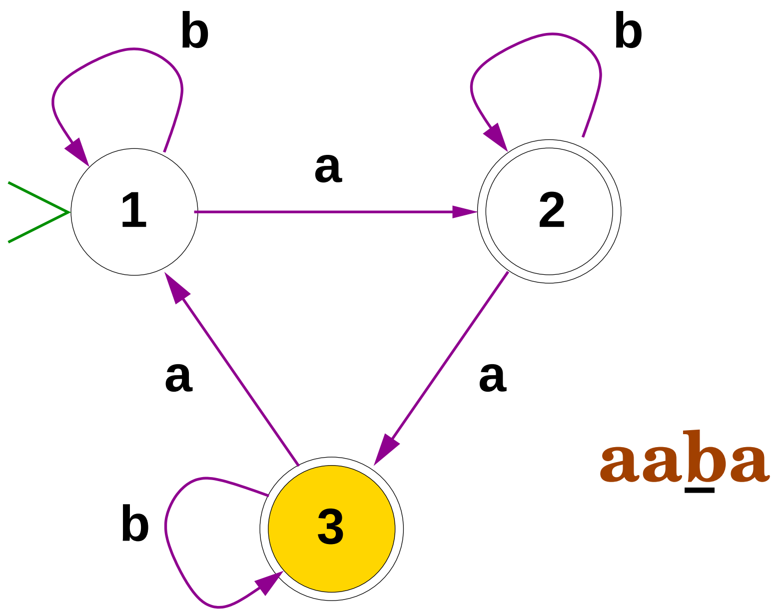
- State 1 (initial). Nothing read yet.

A non-accepted string



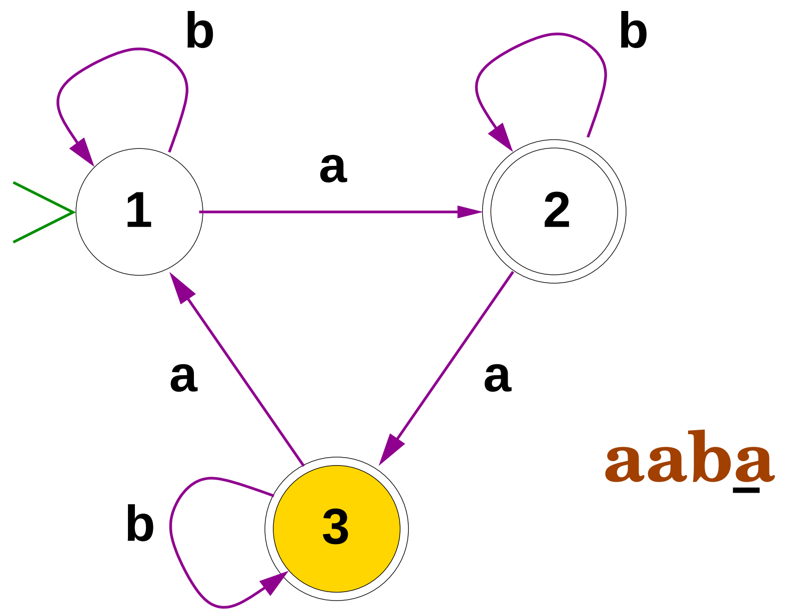
- Read a, State 2.

A non-accepted string



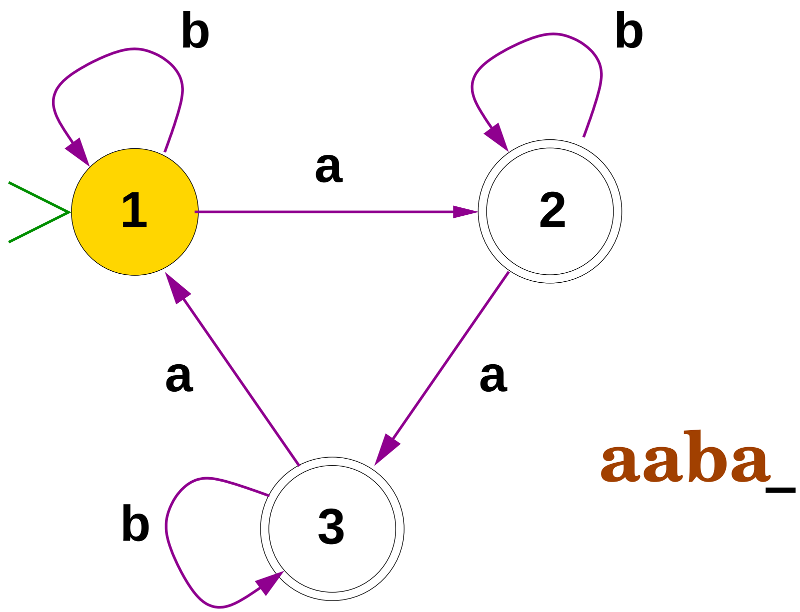
- Read aa, state 3.

A non-accepted string



- Read aab, state 3.

A non-accepted string



- Finished reading `aaba`, state 1, not accepted.

A computation trace

- For our example above, the computation for the string $\mathtt{baab}$ is

$$1 \xrightarrow{\mathtt{b}} 1 \xrightarrow{\mathtt{a}} 2 \xrightarrow{\mathtt{a}} 3 \xrightarrow{\mathtt{b}} 3.$$

  Abbreviated notation: $1 \xrightarrow{\mathtt{baab}} 3$

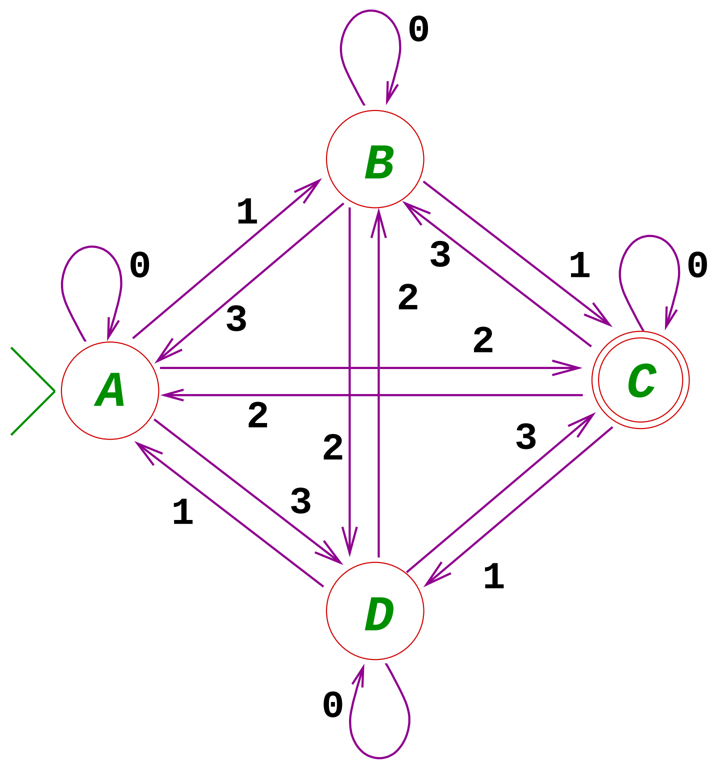
- The computation for the string $\mathtt{aaba}$ is

$$1 \xrightarrow{\mathtt{a}} 2 \xrightarrow{\mathtt{a}} 3 \xrightarrow{\mathtt{b}} 3 \xrightarrow{\mathtt{a}} 1.$$

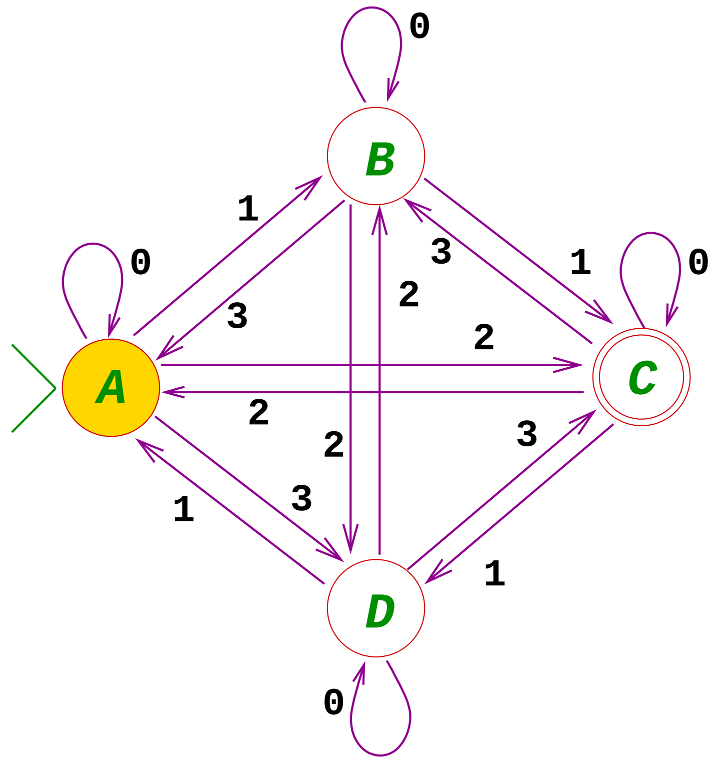
  Abbreviated notation: $1 \xrightarrow{\mathtt{aaba}} 3$

Example: Addition mod 4

- The following automaton is over the alphabet $\{0, 1, 2, 3\}$
- It accept a string of digits iff they add up to 2 modulo 4.

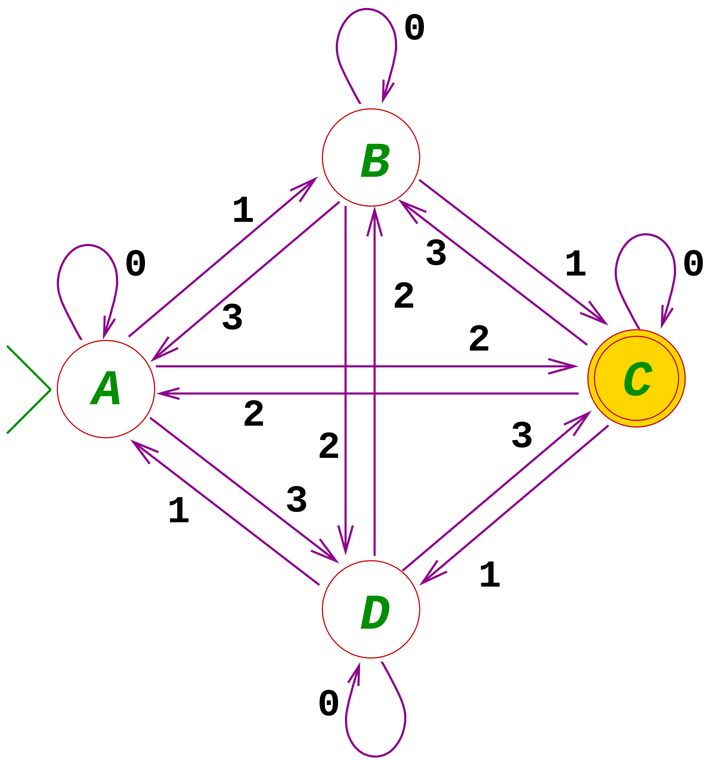


- Reading input 21032 from initial state A:



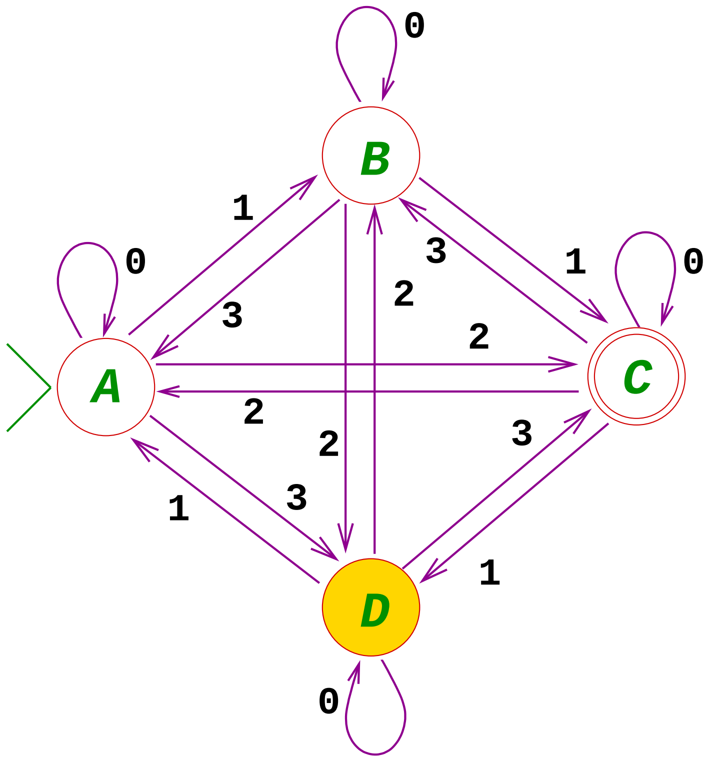
A 21032

- Reads remaining string 1032:



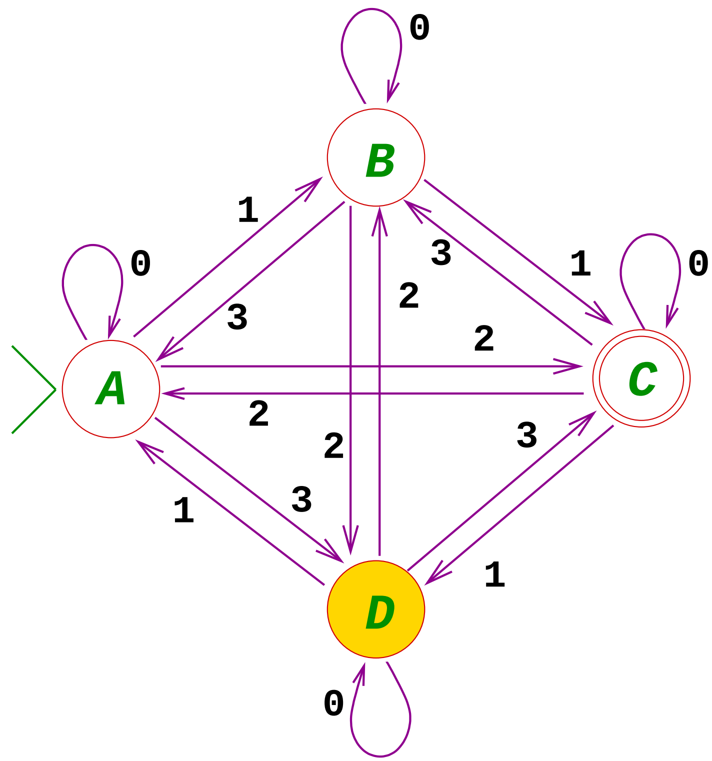
C **1032**

- Reads remaining string 032:



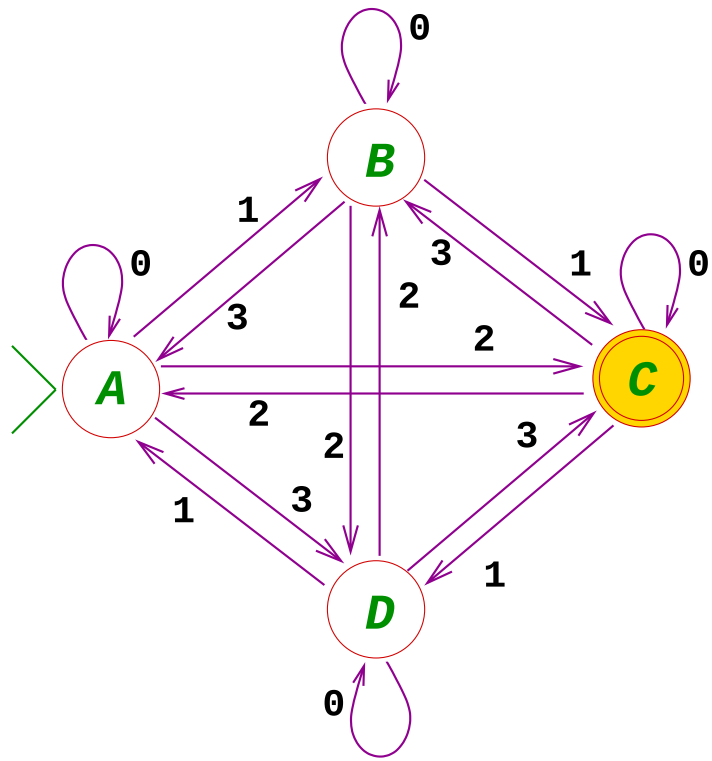
D **032**

- Reads remainder 32:



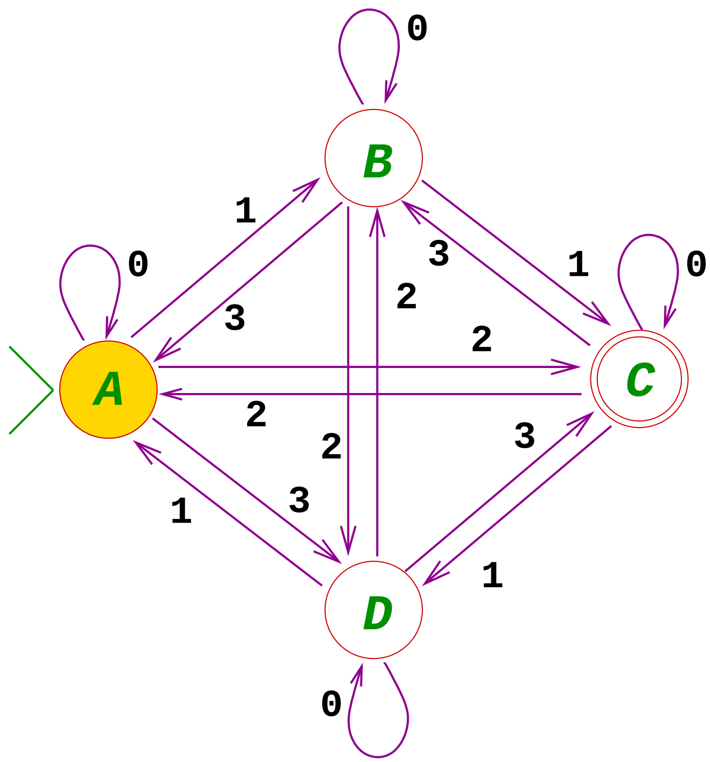
D **32**

• Reads remainder 2:



C 2

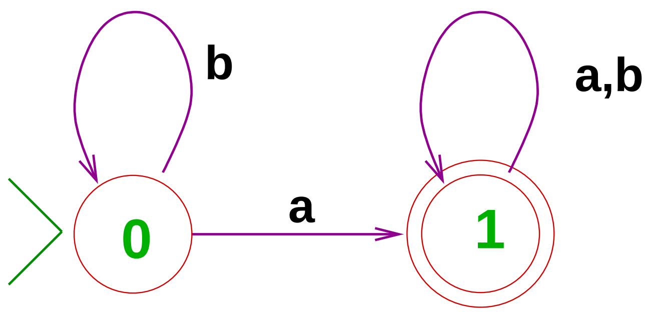
• Reads remainder ε (empty string):



A ε

• Ends reading. A not an accept-state, 21032 not accepted.

Additional examples

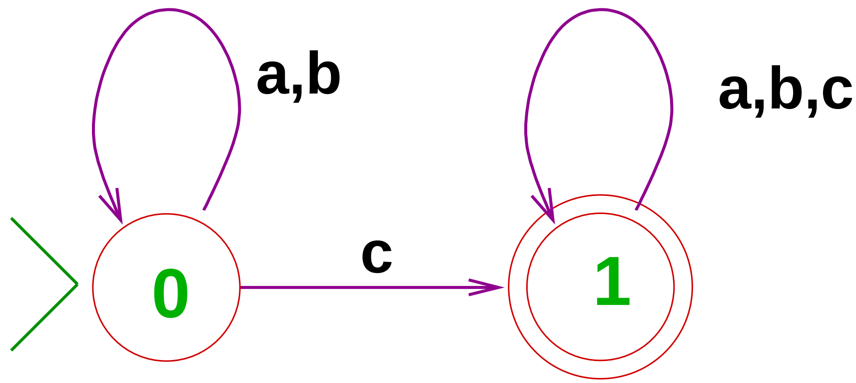


$0 \xrightarrow{\texttt{b}} 0 \xrightarrow{\texttt{a}} 1 \xrightarrow{\texttt{b}} 1 \xrightarrow{\texttt{b}} 1 \xrightarrow{\texttt{a}} 1$

$0 \xrightarrow{\texttt{b}} 0 \xrightarrow{\texttt{b}} 0 \xrightarrow{\texttt{b}} 0 \xrightarrow{\texttt{b}} 0$

What is the language recognized?

Three letter example

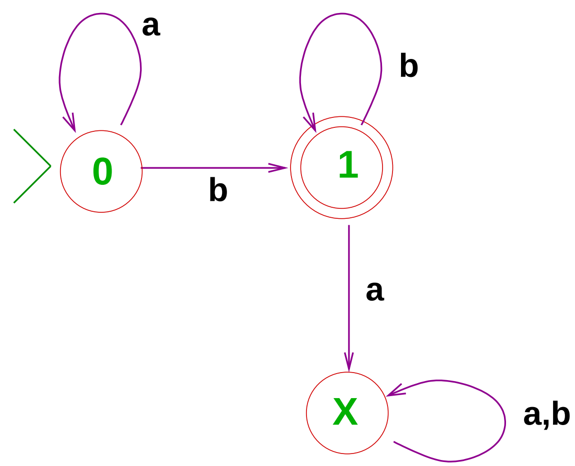


$0 \xrightarrow{\texttt{a}} 0 \xrightarrow{\texttt{b}} O \xrightarrow{\texttt{a}} 0 \xrightarrow{\texttt{c}} 1 \xrightarrow{\texttt{b}} 1$

$0 \xrightarrow{\texttt{c}} 1 \xrightarrow{\texttt{b}} 1 \xrightarrow{\texttt{a}} 1 \xrightarrow{\texttt{b}} 1 \xrightarrow{\texttt{a}} 1$

What are the language accepted?

An automaton with a sink



$0 \xrightarrow{\mathtt{a}} 0 \xrightarrow{\mathtt{a}} 0 \xrightarrow{\mathtt{b}} 1 \xrightarrow{\mathtt{b}} 1 \xrightarrow{\mathtt{b}} 1$

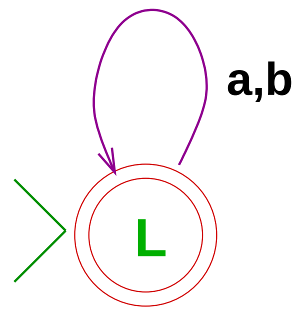
$0 \xrightarrow{\mathtt{b}} 1 \xrightarrow{\mathtt{b}} 1 \xrightarrow{\mathtt{a}} X \xrightarrow{\mathtt{b}} X \xrightarrow{\mathtt{a}} X$

Note: Every state has exactly one arrow for every $\sigma \in \Sigma$.

- A ***sink*** is a non-accepting state with all outgoing transitions pointing to itself.

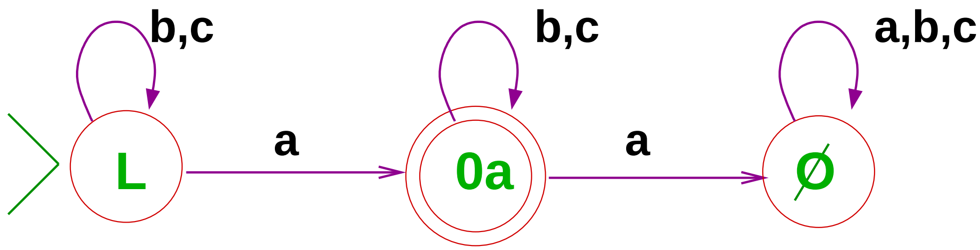
Example

Here is a trivial automaton with a single state:



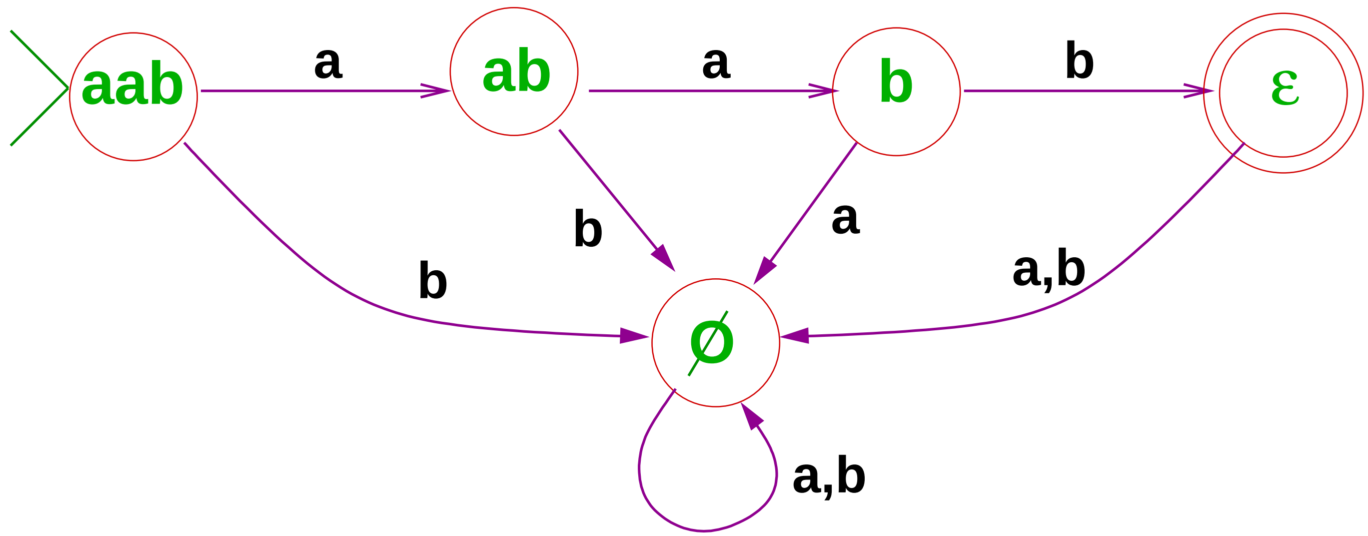
What strings are accepted?

Example



accepts the strings with exactly one `a`, and no other.

Example



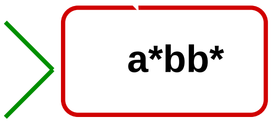
accepts the string `aab` and no other.

CONSTRUCTING AUTOMATA

From a language to arecognizing automaton

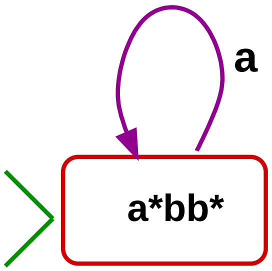
- We give a method that, given a language L,
  attempts to construct a DFA M recognizing L.
- If and when the process teminates, we obtain such an M.
- We start with a couple of non-trivial examples,
  before articulating the method and giving more examples.

Example: a's precede b's



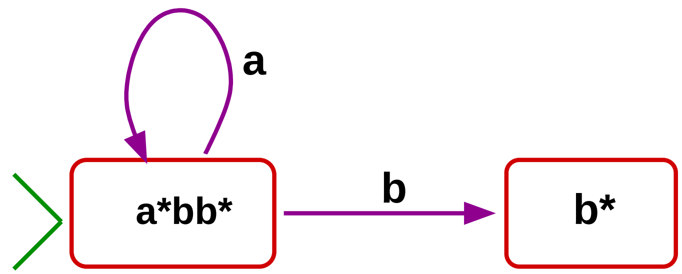
- Construct an automaton recognizing $\mathcal{L}(a^*bb^*)$. That is, accepting strings of a's followed by one or more b's, and ***only*** those.
- The initial state is the declaration of this goal.
- What will be an updated goal after reading an a ?

Reading an `a`



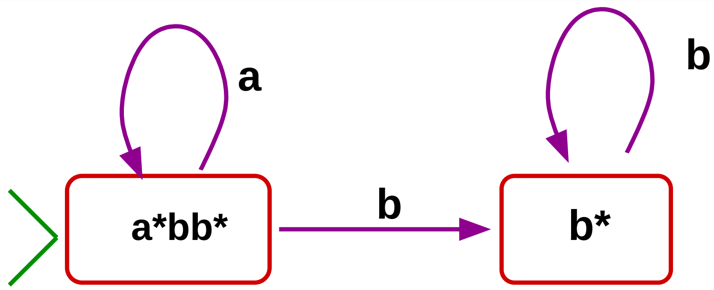
- The goal is unchanged!.
- But what happens if we read a `b` ?

Reading a `b`



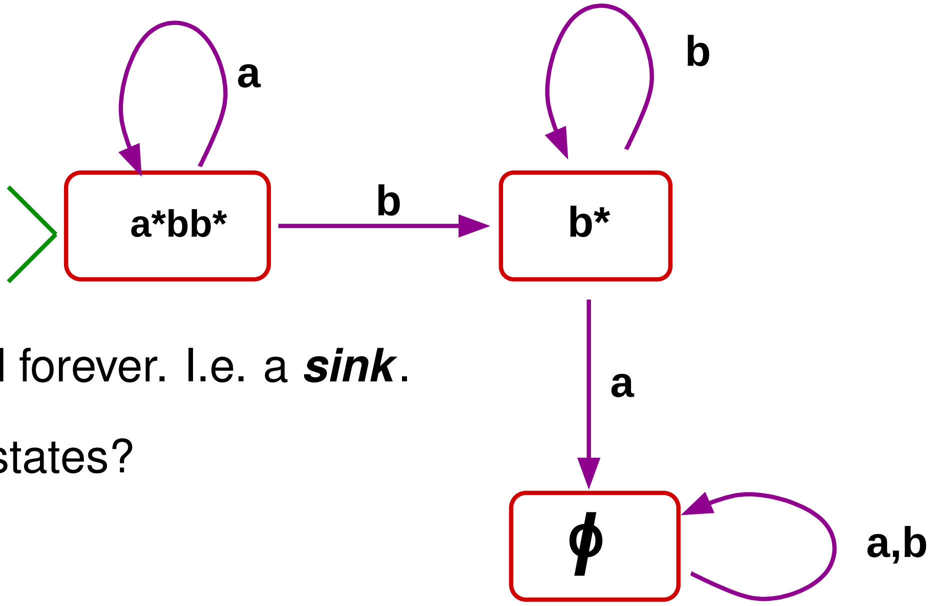
- A new goal: from now on only `b`'s, any number.
- What if we read a `b` ***now***?

Reading another b



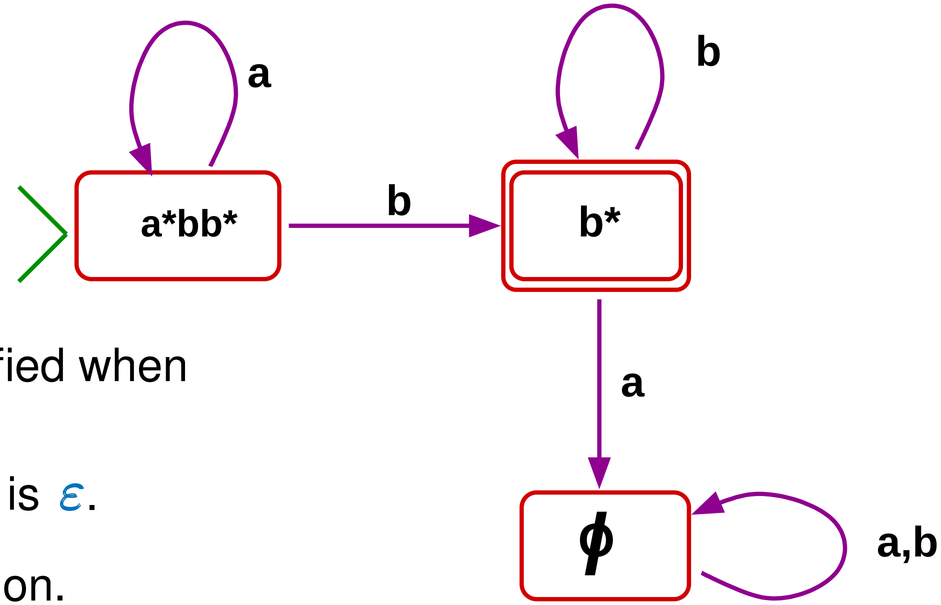
- No change.
- And what if, instead, we read an a ?

Reading an `a` instead



- This is a non-accept, now and forever. I.e. a ***sink***.
- And which are the accepting states?

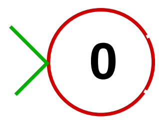
What are the accepting states



- Accept if current goal is satisfied when
  nothing left to read,
  i.e. when the current string is ε.
- This completes the construction.

Example: Ending as it starts

0 $\sigma w \sigma$

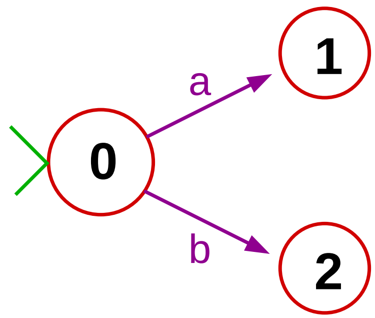


*

- Construct an automaton accepting strings $\sigma w \sigma$,
  i.e. with last letter identical to the first, and ***no others***.
- The initial state is the declaration of this goal.
- What will be the updated goals after reading the first letter?

Example: Ending as it starts

Reading the first letter:



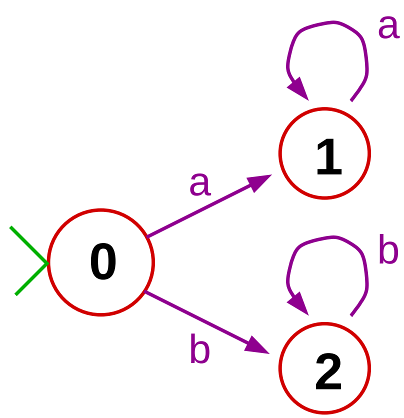
0	$\sigma w \sigma$
1	ε **|** wa
2	**|** wb

*

- Either this is the last letter, or else it repeats at the end.
- What if we now read this letter again?

Example: Ending as it starts

Sought letter repeated:



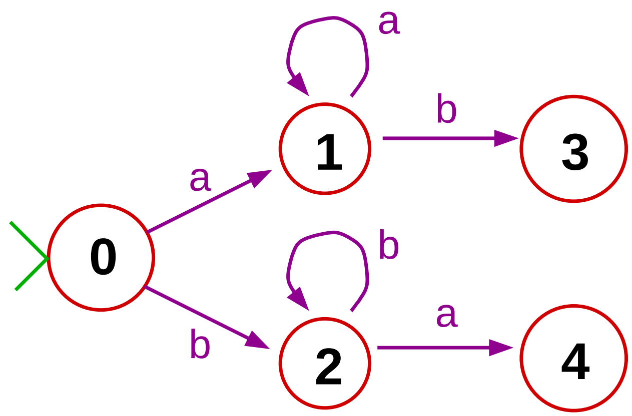
0	$\sigma w \sigma$
1	ε **|** wa
2	ε **|** wb

*****

- The goal does not change.
- And what about the opposite letter ***now***?

Example: Ending as it starts

Reading opposite letter:



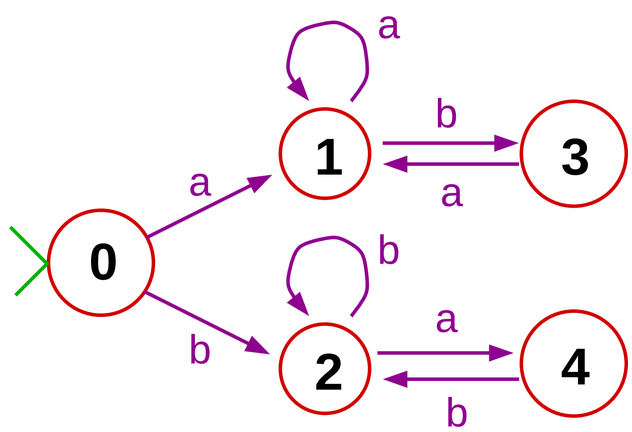
0	$\sigma w \sigma$
1	ε **|** wa
2	ε **|** wb
3	wa
4	wb

*

- The option of not reading further has been blocked.

Example: Ending as it starts

Opposite letter repeating:



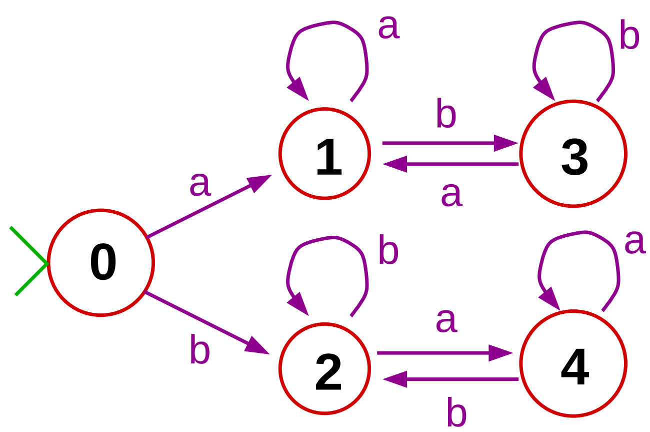
0	$\sigma w \sigma$
1	ε | wa
2	ε | wb
3	wa
4	wb

*

- But if the sought letter is read now,
  the previous goal is restored.
- And if we keep reading the wrong letter?

Example: Ending as it starts

Return to sought letter:



0 $\sigma w \sigma$

1 ε **|** wa

2 ε **|** wb

3 wa

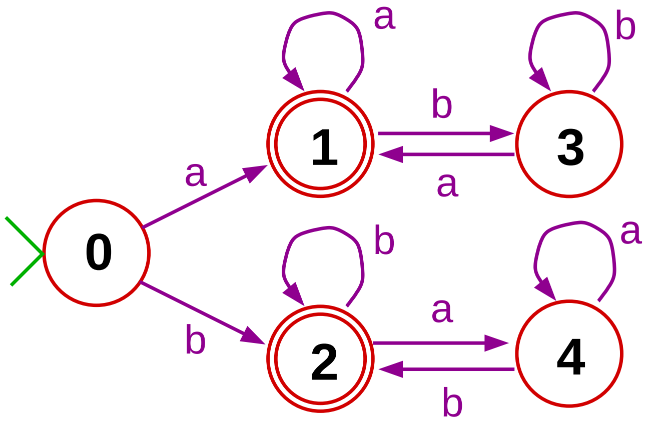
4 wb

*

- No change of goal.
- What are the accepting states?

Example: Ending as it starts

The accepting states:



0	$\sigma w \sigma$
1	ε **|** wa
2	ε **|** wb
3	wa
4	wb

*

- Accept if current goal is satisfied when nothing left to read.
- This completes the construction.

Goal oriented automaton construction

- *When you head to an unfamiliar destination,*
  would you prefer the GPS map to display the road already covered,
  or rather the road ahead?

Goal oriented automaton construction

- *When you head to an unfamiliar destination,*
  would you prefer the GPS map to display the road already covered,
  or rather the road ahead?

- Programming is a ***goal oriented*** process.
  The relevant mission is to achieve a goal.
  The initial task of an acceptor for L is
  "accept the strings in L ***and no others"!***

Goal oriented automaton construction

- *When you head to an unfamiliar destination,*
  would you prefer the GPS map to display the road already covered,
  or rather the road ahead?

- Programming is a ***goal oriented*** process.
  The relevant mission is to achieve a goal.
  The initial task of an acceptor for L is
  "accept the strings in L ***and no others"!***

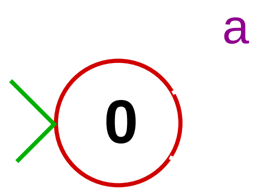
- The tasks are adjusted as the input string is read.
  Each task is of the form

the string ahead leads into a string in L

Identifying accepting tasks

- The development above updates states (conditions) as required when symbols σ are read.
- A string $x = \sigma u$ satisfying the current condition (=state) leads to A iff u started at the next condition leads to A.
- So the accepting conditions are the ones that are satisfied when reading ends, i.e. when the string-ahead is ε.

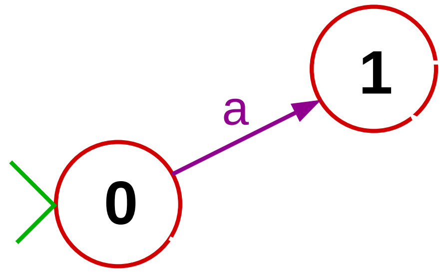
Example: Repeated last symbol



state dictionary

0 $w\sigma\sigma$

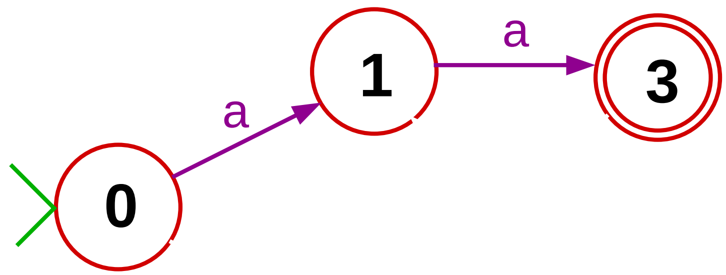
Example: Repeated last symbol



0 $w\sigma\sigma$

1 a **|** $w\sigma\sigma$

Example: Repeated last symbol

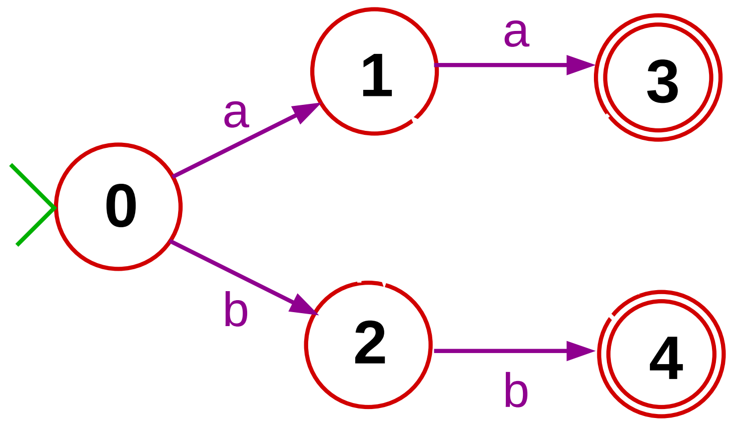


0 $w\sigma\sigma$

1 a **|** $w\sigma\sigma$

3 ε **|** a **|** $w\sigma\sigma$

Example: Repeated last symbol



0 $w\sigma\sigma$

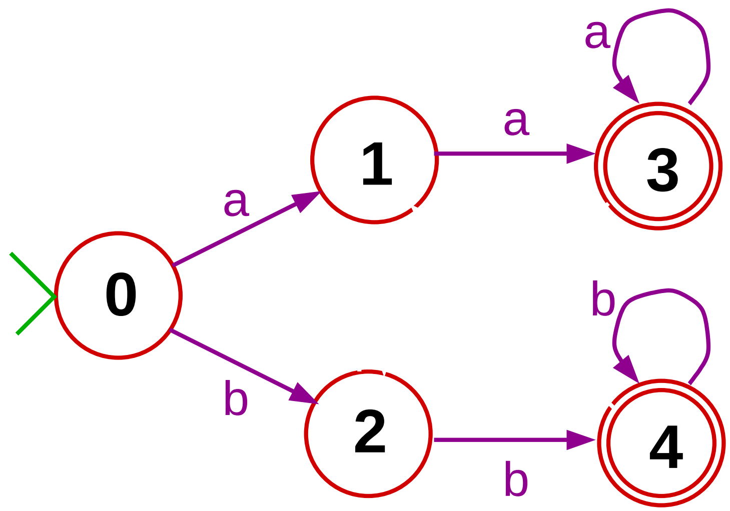
1 a | $w\sigma\sigma$

2 b | $w\sigma\sigma$

3 ε | a | $w\sigma\sigma$

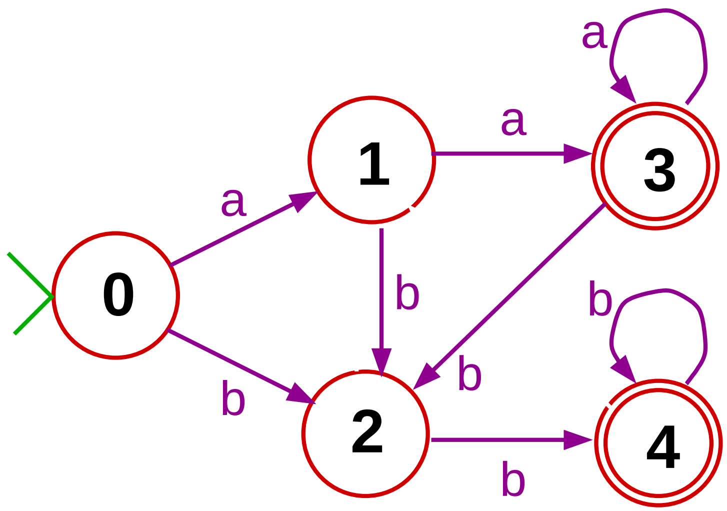
4 ε | b | $w\sigma\sigma$

Example: Repeated last symbol



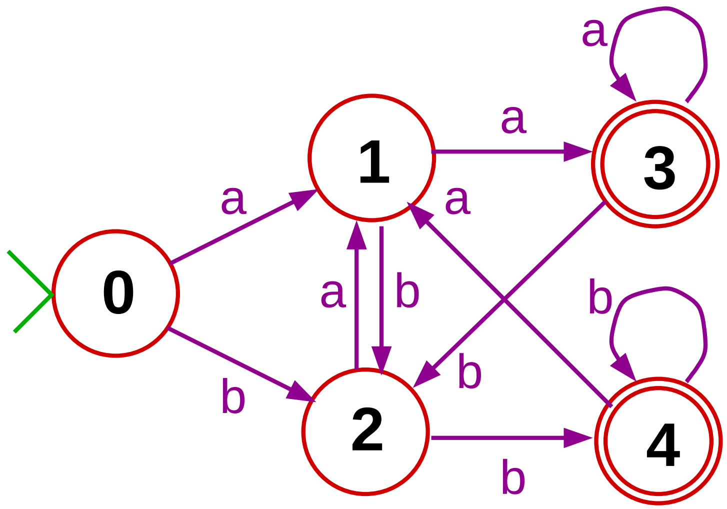
0	$w\sigma\sigma$		
1	a	$w\sigma\sigma$	
2	b	$w\sigma\sigma$	
3	ε	a	$w\sigma\sigma$
4	ε	b	$w\sigma\sigma$

Example: Repeated last symbol



0	$w\sigma\sigma$
1	a **|** $w\sigma\sigma$
2	b **|** $w\sigma\sigma$
3	ε **|** a **|** $w\sigma\sigma$
4	ε **|** b **|** $w\sigma\sigma$

Example: Repeated last symbol



0 $w\sigma\sigma$

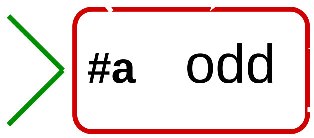
1 a | $w\sigma\sigma$

2 b | $w\sigma\sigma$

3 ε | a | $w\sigma\sigma$

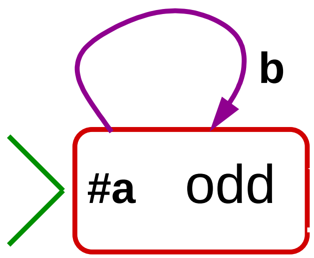
4 ε | b | $w\sigma\sigma$

Example: Recognizing odd length



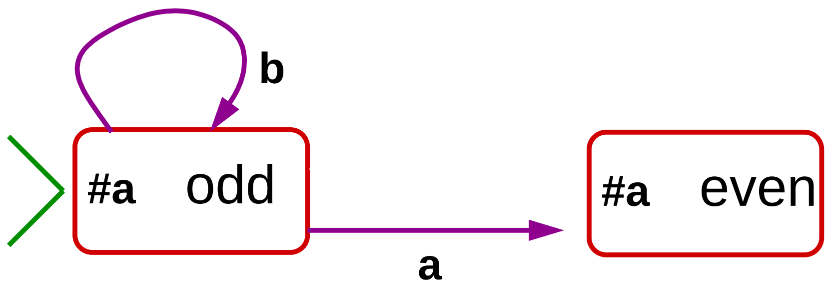
- Initial task: accept strings with an odd number of `a`'s

Example: Recognizing odd length



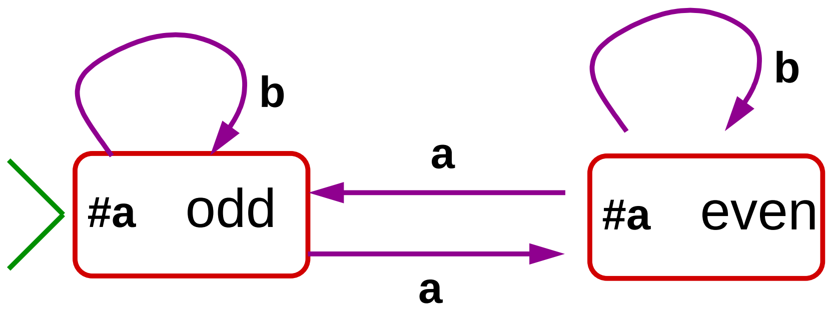
- Reading a `b` does not change the task

Example: Recognizing odd length



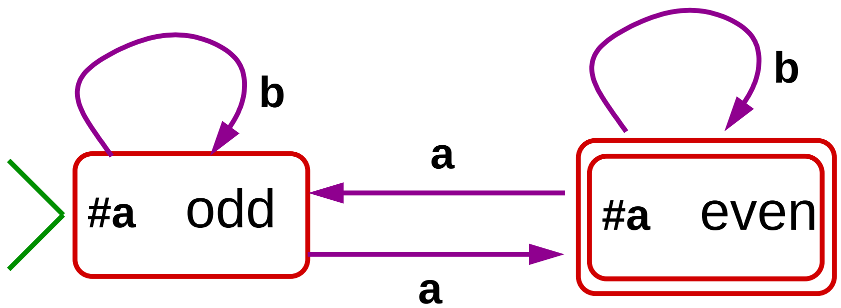
- Reading an a revises the task to:
  accept strings with an even number of a's

Example: Recognizing odd length



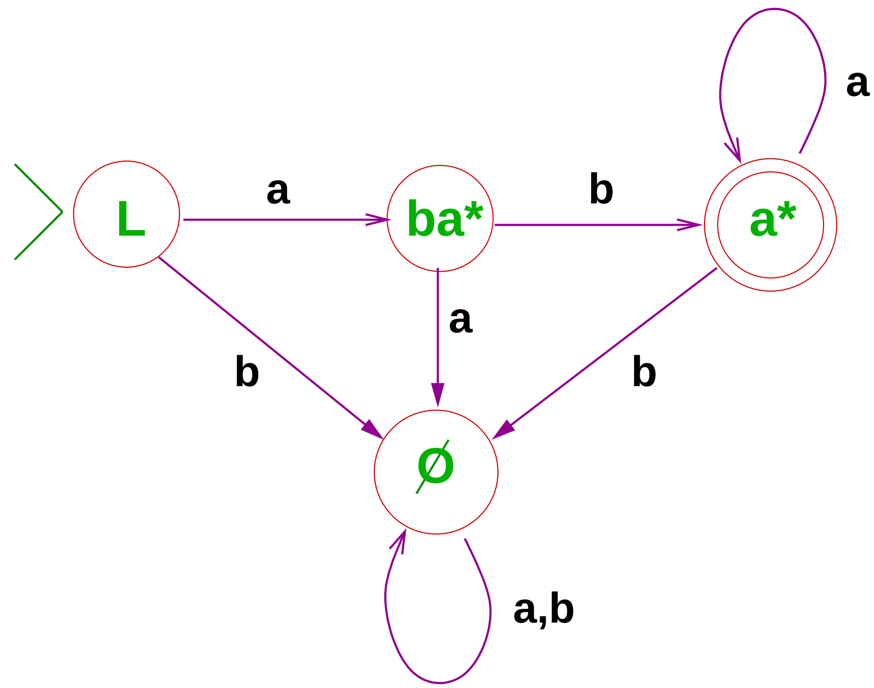
- Same reasoning for the “even” task

Example: Recognizing odd length



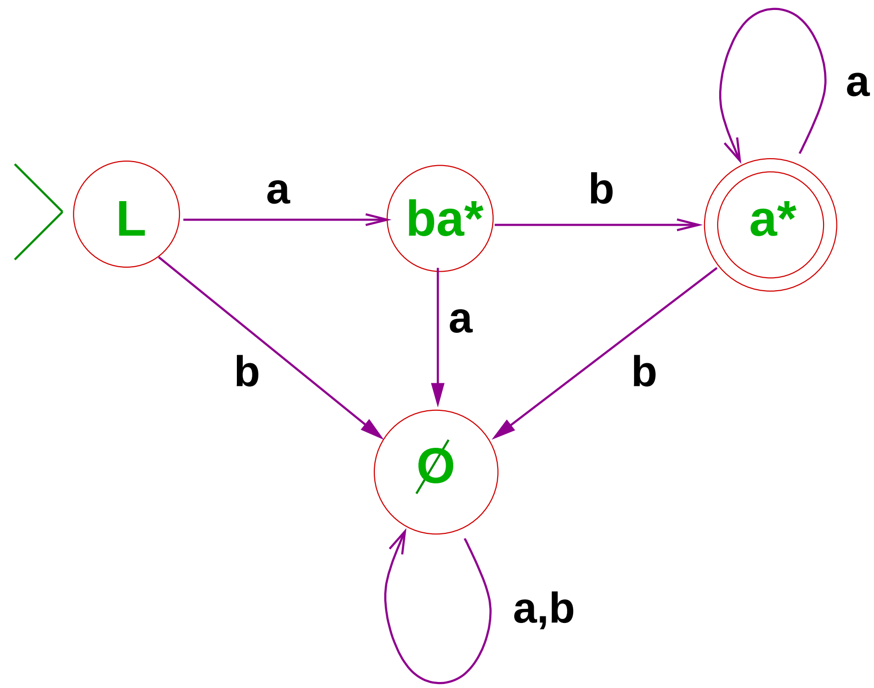
- Accept description fulfilled by ε.

Example: aba*



Accepts the strings of the form $\mathtt{aba}^n$ with $n \geqslant 0$, and no others.

Example: aba*

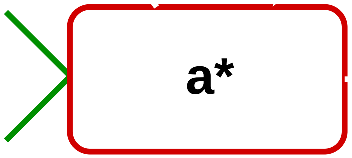


Accepts the strings of the form $\mathtt{aba}^n$ with $n \geqslant 0$, and no others.

- Note the sink at the bottom of the diagram.

A trivial example: Just $\mathtt{a}$ *'s*

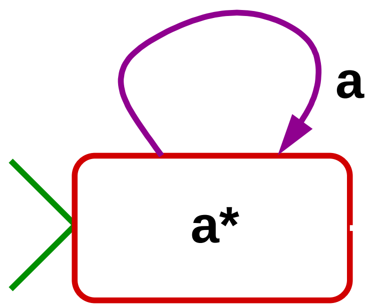
Construct an automaton recognizing $\mathcal{L}(\mathtt{a}^*)$
as a sub-language of $\{\mathtt{a}, \mathtt{b}\}^*$



- Initial task: accept strings of $\mathtt{a}$'s

A trivial example: Just a *'s*

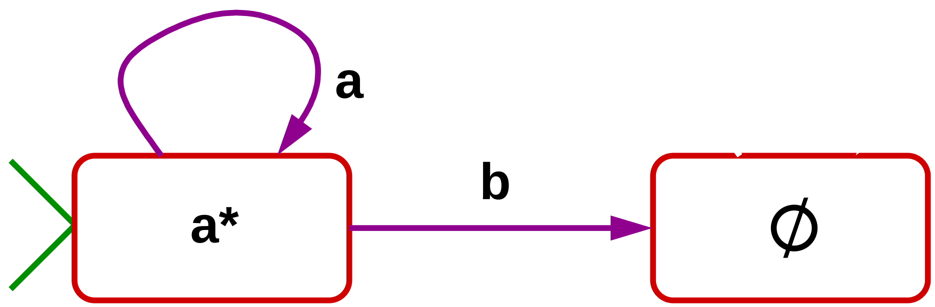
Construct an automaton recognizing $\mathcal{L}(\mathtt{a}^*)$ as a sub-language of $\{\mathtt{a}, \mathtt{b}\}^*$



- Reading an a does not change the task

A trivial example: Just $\texttt{a}$ *'s*

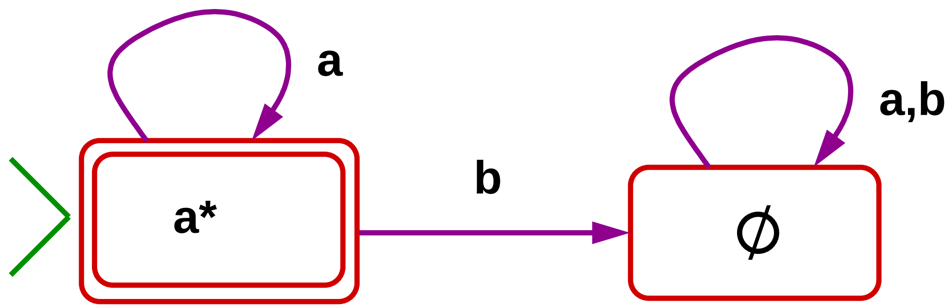
Construct an automaton recognizing $\mathcal{L}(\texttt{a}^*)$
as a sub-language of $\{\texttt{a},\texttt{b}\}^*$



- Reading a $\texttt{b}$ revises the task to not accepting anything. A ***sink.***

A trivial example: Just a *'s*

Construct an automaton recognizing $\mathcal{L}(\mathtt{a}^*)$
as a sub-language of $\{\mathtt{a}, \mathtt{b}\}^*$

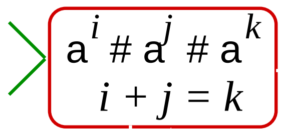


- No escape from the sink

Example: Addition mod 2

Automaton over $\{\mathtt{a}, \#\}$ recognizing

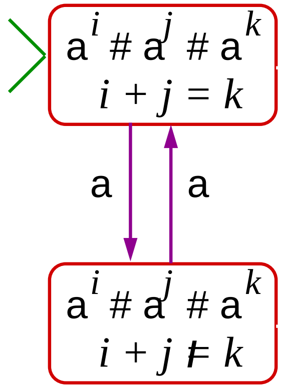
$$\{\mathtt{a}^i \# \mathtt{a}^j \# \mathtt{a}^k \mid i + j = k \mod 2\}$$



Example: Addition mod 2

Automaton over $\{a, \#\}$ recognizing

$$\{a^i \# a^j \# a^k \mid i + j = k \mod 2\}$$

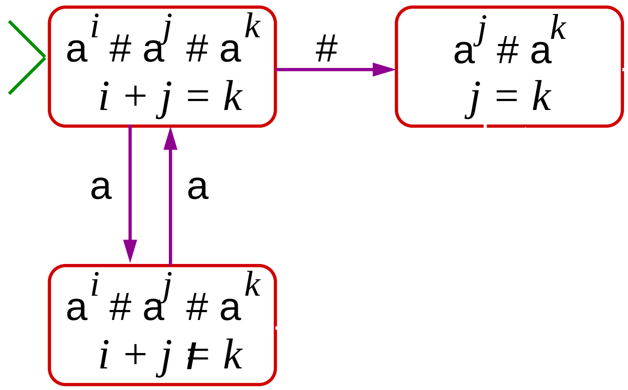


Reading a's toggles between equlity and inequality of parities.

Example: Addition mod 2

Automaton over $\{\mathsf{a}, \#\}$ recognizing

$$\{\mathsf{a}^i \# \mathsf{a}^j \# \mathsf{a}^k \mid i + j = k \mod 2\}$$

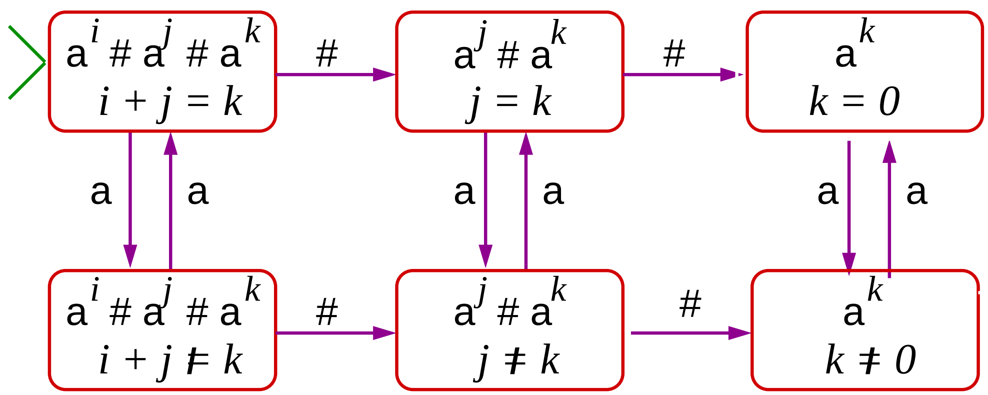


Reading the separator $\#$ means $i = 0$.

Example: Addition mod 2

Automaton over $\{\mathrm{a}, \#\}$ recognizing

$$\{\mathrm{a}^i \# \mathrm{a}^j \# \mathrm{a}^k \mid i + j = k \mod 2\}$$

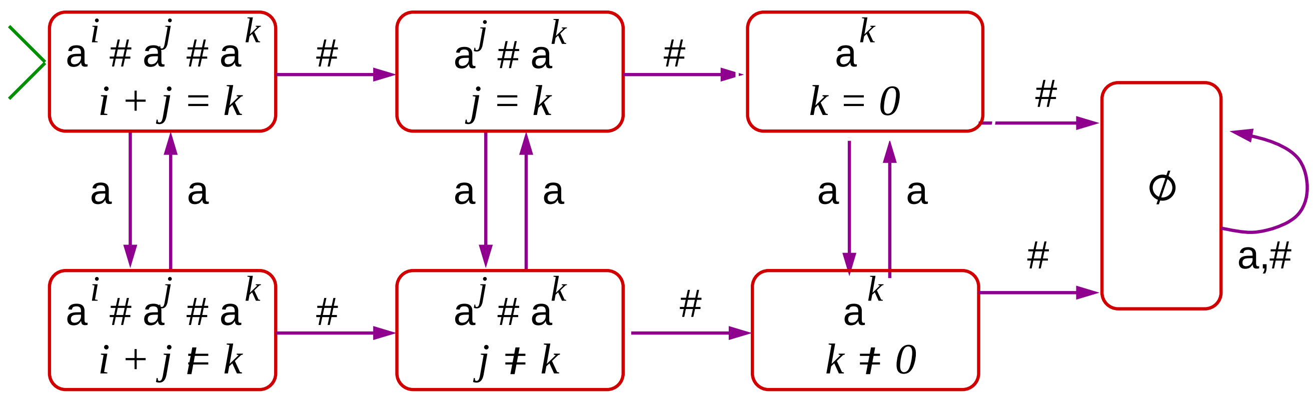


The same arguments are repeated

Example: Addition mod 2

Automaton over $\{a, \#\}$ recognizing

$$\{a^i \# a^j \# a^k \mid i + j = k \mod 2\}$$

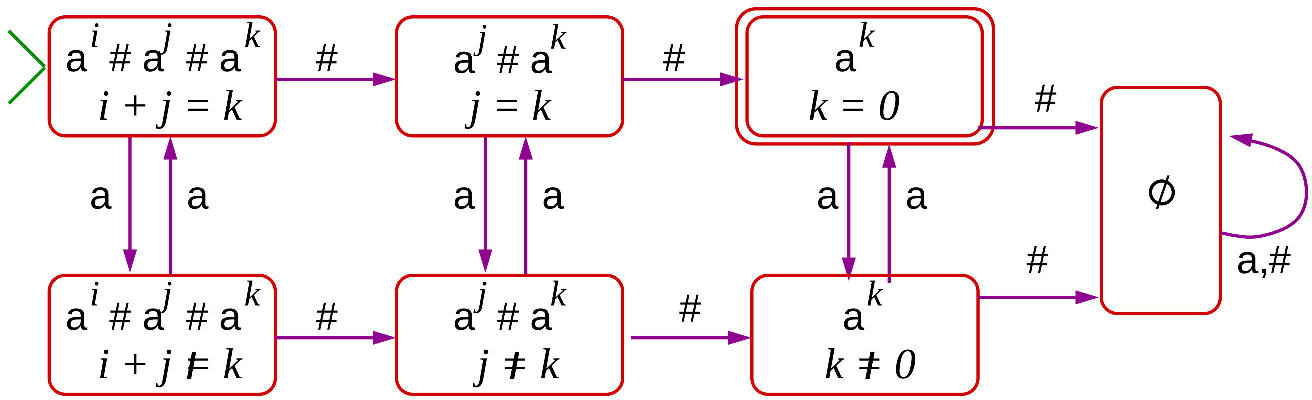


Encountering an extra separator leads to a sink

Example: Addition mod 2

Automaton over $\{\mathsf{a}, \#\}$ recognizing

$$\{\mathsf{a}^i \# \mathsf{a}^j \# \mathsf{a}^k \mid i + j = k \mod 2\}$$



The single one accepting state is the one satisfied by ε.

Summary of the method

- The initial acceptance-condition is
  the language to be recognized.

Summary of the method

- The initial acceptance-condition is
  the language to be recognized.
- Given a new acceptance-condition, each each $\sigma \in \Sigma$
  find what condition is required after reading σ .

Summary of the method

- The initial acceptance-condition is
  the language to be recognized.
- Given a new acceptance-condition, each each $\sigma \in \Sigma$
  find what condition is required after reading σ .
- That is, a string σu satisfies the current condition iff
  u satisfies the condition after σ is read.

Summary of the method

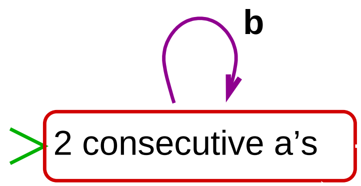
- The initial acceptance-condition is
  the language to be recognized.
- Given a new acceptance-condition, each each $\sigma \in \Sigma$
  find what condition is required after reading σ .
- That is, a string σu satisfies the current condition iff
  u satisfies the condition after σ is read.
- A condition is an ***accepting state*** iff it is satisfied by ε.

Example: Two consecutive $\mathtt{a}$'s

Construct an automaton recognizing $\mathcal{L}(\Sigma^* \cdot \mathtt{aa} \cdot \Sigma^*)$

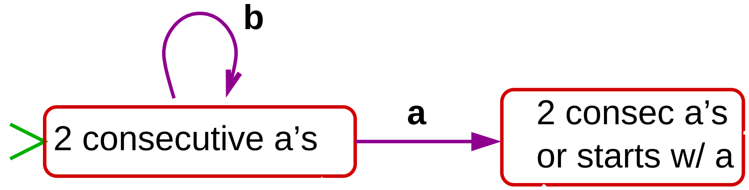
Example: Two consecutive a*'s*

Reading b leaves the task unchanged:



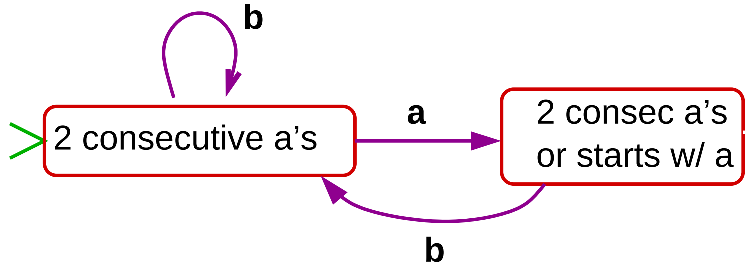
Example: Two consecutive `a` *'s*

But reading `a` opens two future options:



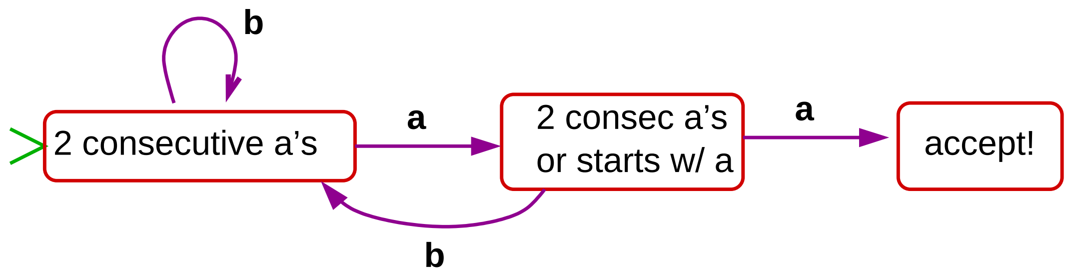
Example: Two consecutive a*'s*

From these two options reading b kills the first:



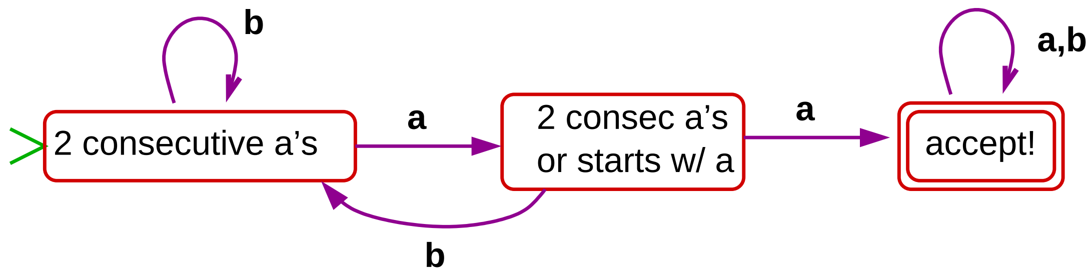
Example: Two consecutive `a`*'s*

But reading an `a` settles acceptance:

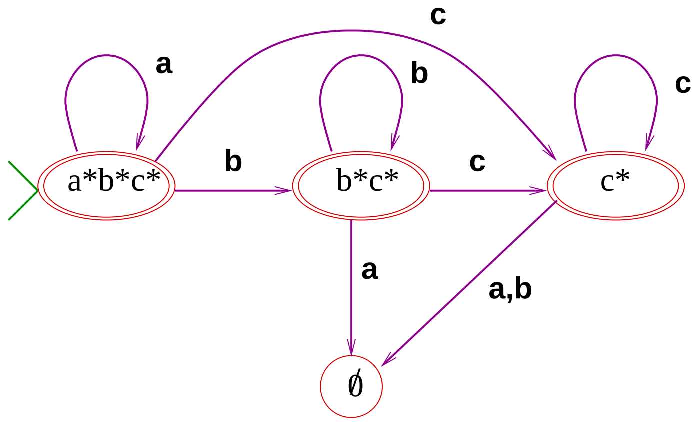


Example: Two consecutive a's

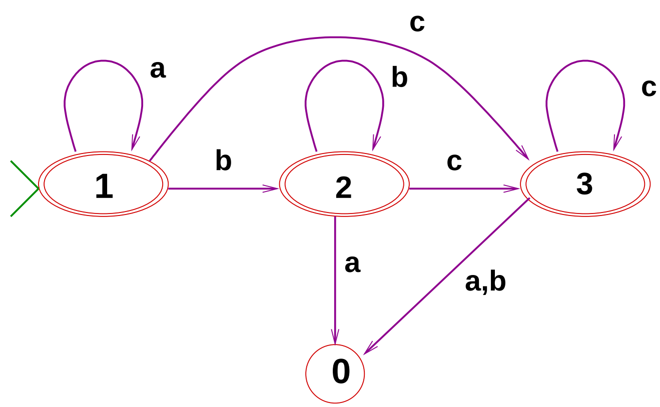
No further reading alterns that conclusion:



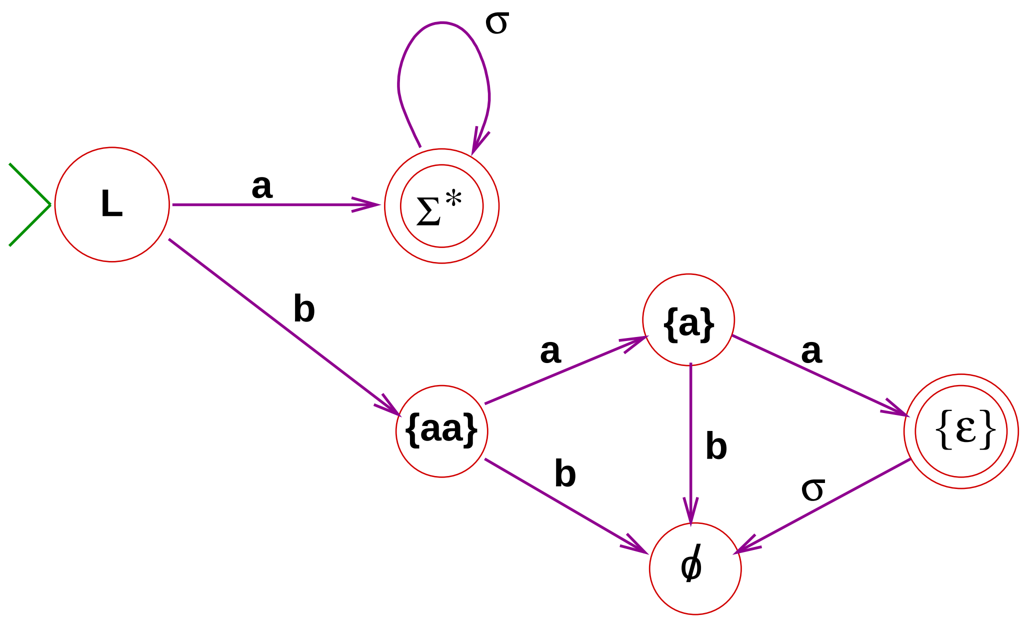
Example 7: `a*b*c*`



- Label states as we wish, with optional “dictionary.”



Example: Initial `a` or the string `baa`



Example: Symbolic binary addition

- The following example illustrates the use of compound data as "symbols" of an alphabet.

- Consider a long addition in binary, such as

$$\begin{array}{rccccc} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{1} & \mathbf{0} \\ + & \mathbf{0} & \mathbf{1} & \mathbf{1} & \mathbf{0} & \mathbf{1} \\ \hline & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{1} \end{array}$$

Example: Symbolic binary addition

- The following example illustrates the use of compound data as “symbols” of an alphabet.

- Consider a long addition in binary, such as

$$\begin{array}{cccccc} & 0 & 0 & 1 & 1 & 0 \\ + & 0 & 1 & 1 & 0 & 1 \\ \hline & 1 & 0 & 0 & 1 & 1 \end{array}$$

- This table does not look like a string.
  But all such tables have height 3 we can consider each column as a “symbol” in the alphabet $\Sigma = \{0,1\}^3$, that is

$$\Sigma^3 = \left\{ \begin{bmatrix}0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}, \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\1\\1\end{bmatrix} \right\}$$

Example: Symbolic binary addition

- The following example illustrates the use of compound data as "symbols" of an alphabet.

- Consider a long addition in binary, such as

$$\begin{array}{cccccc} & 0 & 0 & 1 & 1 & 0 \\ + & 0 & 1 & 1 & 0 & 1 \\ \hline & 1 & 0 & 0 & 1 & 1 \end{array}$$

- This table does not look like a string.
  But all such tables have height 3 we can consider each column as a "symbol" in the alphabet $\Sigma = \{0, 1\}^3$, that is

$$\Sigma^3 = \{ \begin{bmatrix}0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}, \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\1\\1\end{bmatrix} \}$$

- The long addition above can be consrued as the string $\begin{bmatrix}0\\0\\1\end{bmatrix} \begin{bmatrix}0\\1\\0\end{bmatrix} \begin{bmatrix}1\\1\\0\end{bmatrix} \begin{bmatrix}1\\0\\1\end{bmatrix} \begin{bmatrix}0\\1\\1\end{bmatrix}$

An automaton recognizing symbolic binary addition

- Is there an automaton over Σ^3 that recognizes the correct symbolic binary additions?
- Construct an automaton M that accepts strings like

$$\begin{bmatrix}0\\0\\1\end{bmatrix}\begin{bmatrix}1\\1\\1\end{bmatrix}\begin{bmatrix}1\\1\\1\end{bmatrix}\begin{bmatrix}1\\1\\0\end{bmatrix}$$

but not strings like

$$\begin{bmatrix}0\\1\\1\end{bmatrix}\begin{bmatrix}1\\1\\1\end{bmatrix}\begin{bmatrix}1\\1\\0\end{bmatrix}\begin{bmatrix}1\\0\\0\end{bmatrix}$$

An automaton recognizing symbolic binary addition

Start state is the goal that the table ***adds-up***:
remaining columns add up

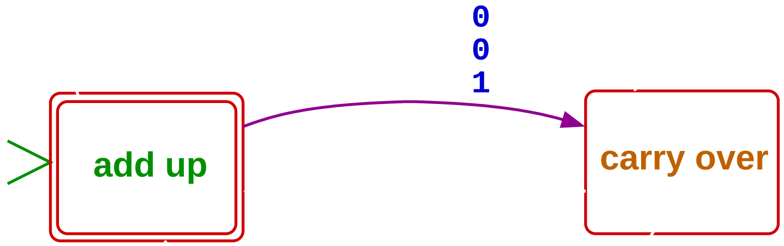
An automaton recognizing symbolic binary addition

Start state is the goal that the table ***adds-up***:

remaining columns add up

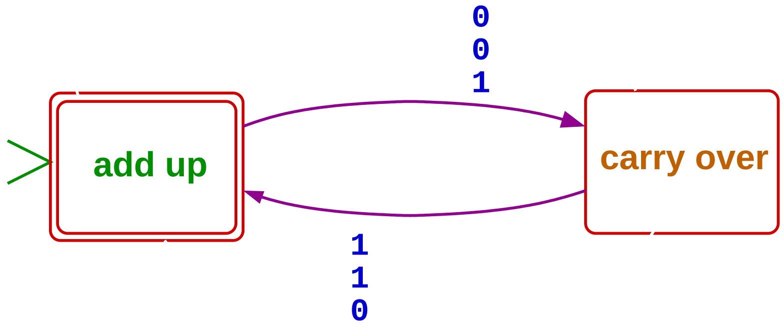
The main other state is *remaining columns yield* ***carry-over***

An automaton recognizing symbolic binary addition



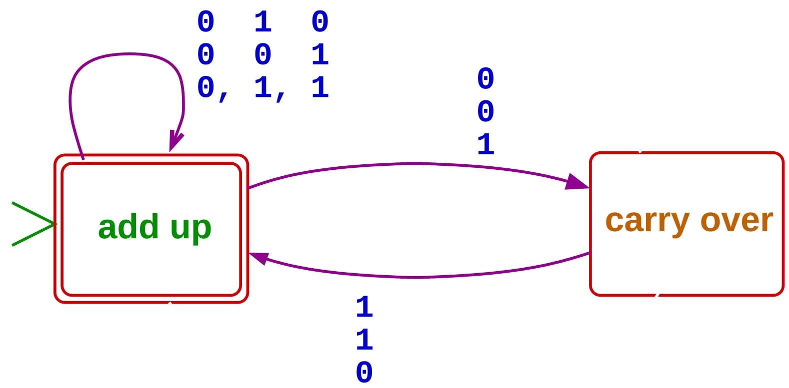
There is one column switching from ***add-up*** to ***carry-over***

An automaton recognizing symbolic binary addition



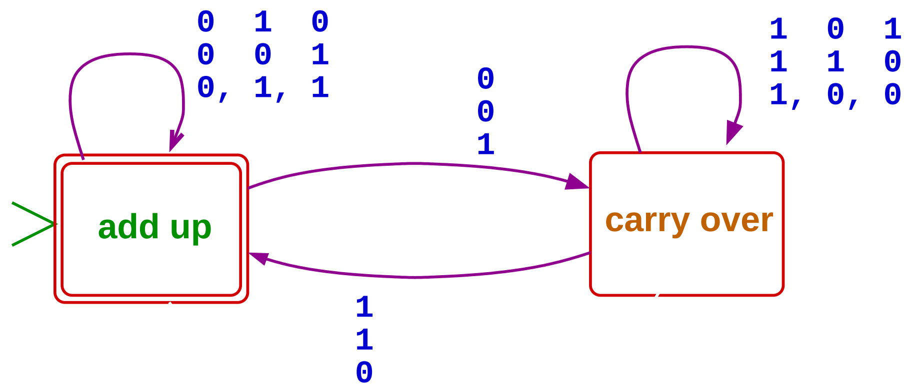
There is one column switching from ***add-up*** to ***carry-over***
and one column switching back from ***carry-over*** to ***add-up***

An automaton recognizing symbolic binary addition



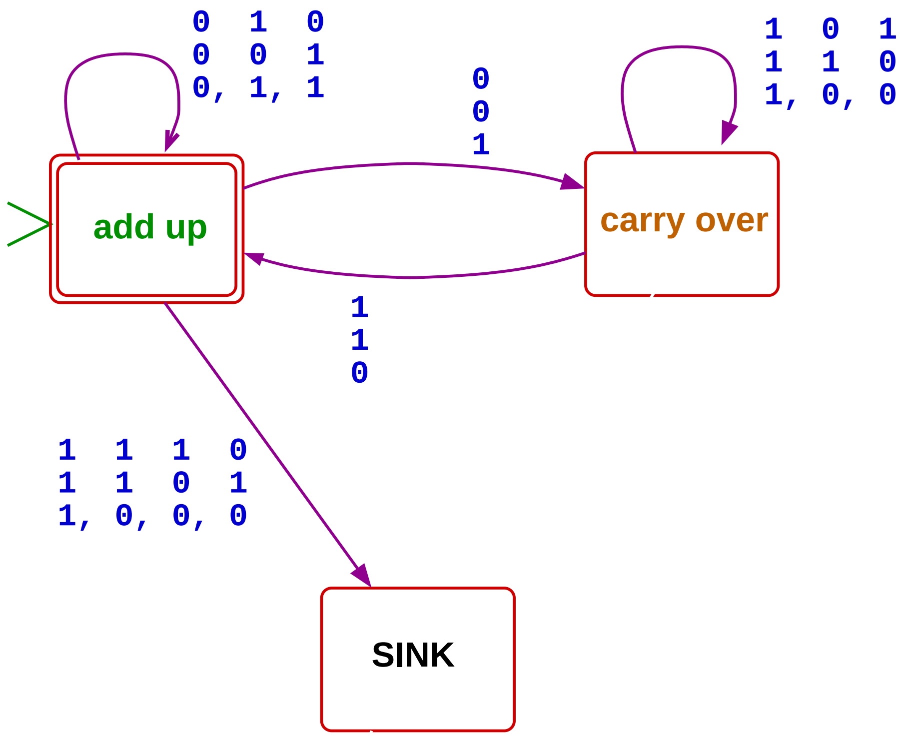
Three columns leave the ***add-up*** goal unchanged

An automaton recognizing symbolic binary addition



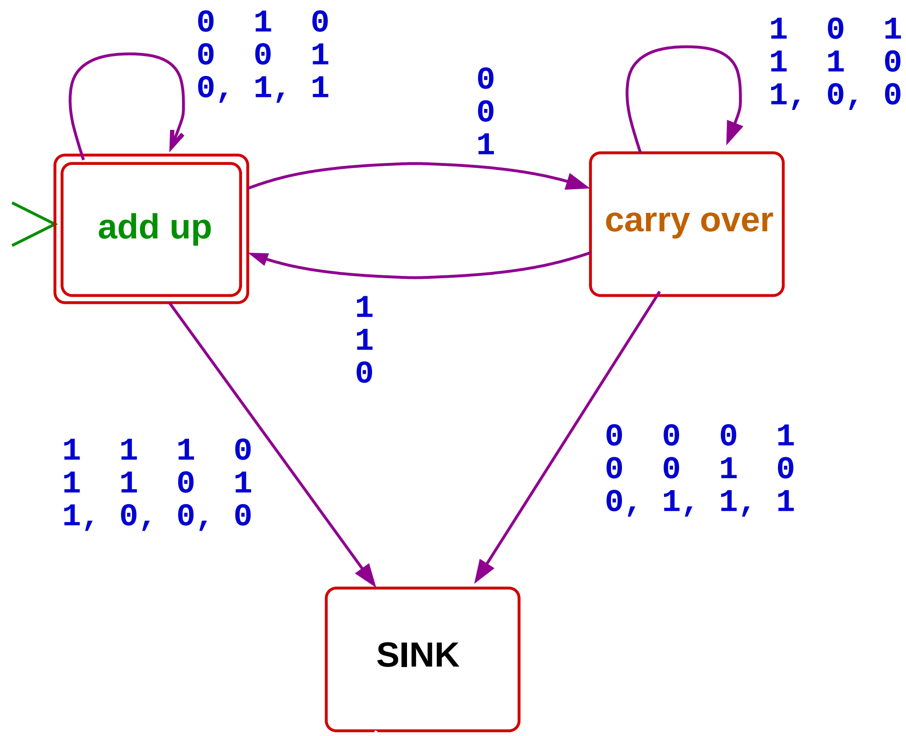
Three columns leave the ***add-up*** goal unchanged
and three leaave ***carry-over*** unchaged

An automaton recognizing symbolic binary addition



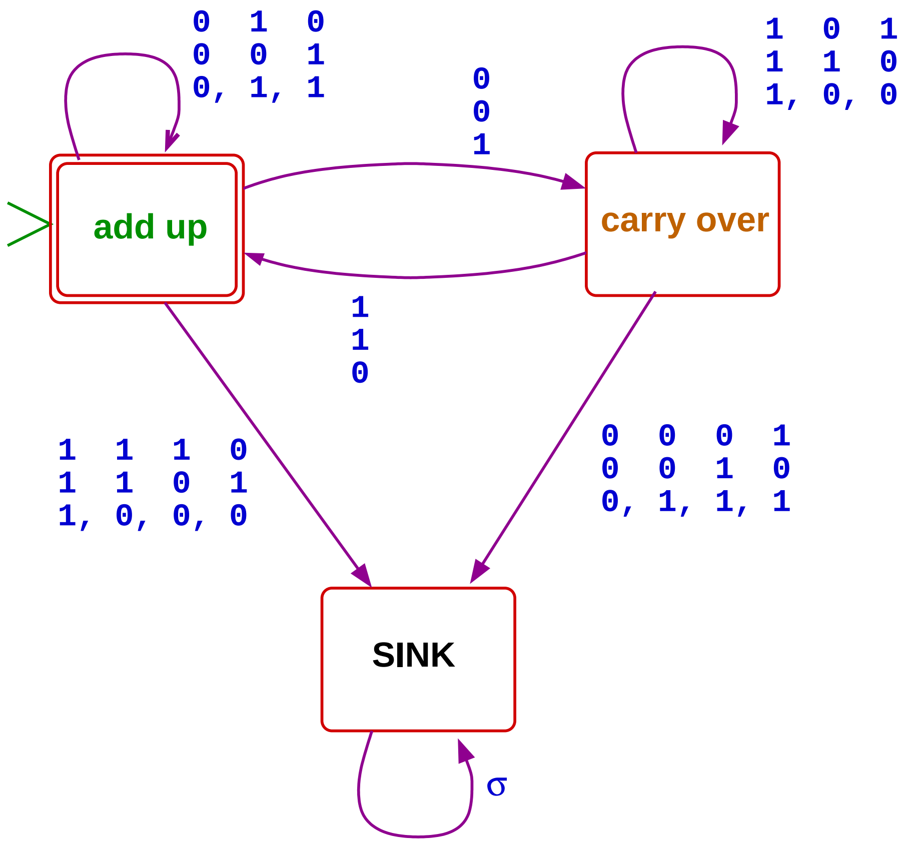
Four columns lead from ***add-up*** to a ***sink***

An automaton recognizing symbolic binary addition



Four columns lead from ***add-up*** to a ***sink***
and four from ***carry-over*** to that ***sink***

An automaton recognizing symbolic binary addition



Finally, ***sink*** is a sink.

Example: Binary numerals divisible by 3

- Consider every string $w \in \{0, 1\}^*$ to be a binary numerals.
- The ***numeric value*** $[w]_2$ of a string $w = d_k d_{k-1} \cdots d_0$ is $\Sigma_i\ 2^i$.
- The numerals divisible by 2 are those that end with 0.

Example: Binary numerals divisible by 3

- Consider every string $w \in \{0, 1\}^*$ to be a binary numerals.
- The ***numeric value*** $[w]_2$ of a string $w = d_k d_{k-1} \cdots d_0$ is $\Sigma_i \ 2^i$.
- Problem: Construct a DFA over $\{0, 1\}^*$ that accepts the numerals divisible by 3.

Example: Binary numerals divisible by 3

- Consider every string $w \in \{0, 1\}^*$ to be a binary numerals.
- The ***numeric value*** $[w]_2$ of a string $w = d_k d_{k-1} \cdots d_0$ is $\Sigma_i \; 2^i$.
- Problem: Construct a DFA over $\{0, 1\}^*$ that accepts the numerals divisible by 3.
- Preliminary: What is the value mod(3) of the digits, i.e. what is 2^k mod(3).

Example: Binary numerals divisible by 3

- Consider every string $w \in \{0, 1\}^*$ to be a binary numerals.
- The ***numeric value*** $[w]_2$ of a string $w = d_k d_{k-1} \cdots d_0$ is $\Sigma_i\ 2^i$.
- Problem: Construct a DFA over $\{0, 1\}^*$ that accepts the numerals divisble by 3.
- Preliminary: What is the value mod(3) of the digits, i.e. what is 2^k mod(3).

  We have that $4^k =_3 1$, by induction on k.

  - $4^0 = 1$
  - If $4^k = 3x + 1$ then $4^{k+1} = 4(3x + 1) = 13x + 1.$

Example: Binary numerals divisible by 3

- Consider every string $w \in \{0,1\}^*$ to be a binary numerals.
- The ***numeric value*** $[w]_2$ of a string $w = d_k d_{k-1} \cdots d_0$ is $\Sigma_i \; 2^i$.
- Problem: Construct a DFA over $\{0,1\}^*$ that accepts the numerals divisble by 3.
- Preliminary: What is the value mod(3) of the digits, i.e. what is 2^k mod(3).

  We have that $4^k =_3 1$, by induction on k.

  So $2^{2k} = 3x + 1$ for some x , and $2^{2k+1} = 2(3x+1) = 6x + 2$.

  $\therefore$ $2^n =_3 1$ for even n , and $=_3 2$ for odd n .

Example: Binary numerals divisible by 3

- For any input w the expectation depends on the parity of $|w|$, the goals are therefore of the form

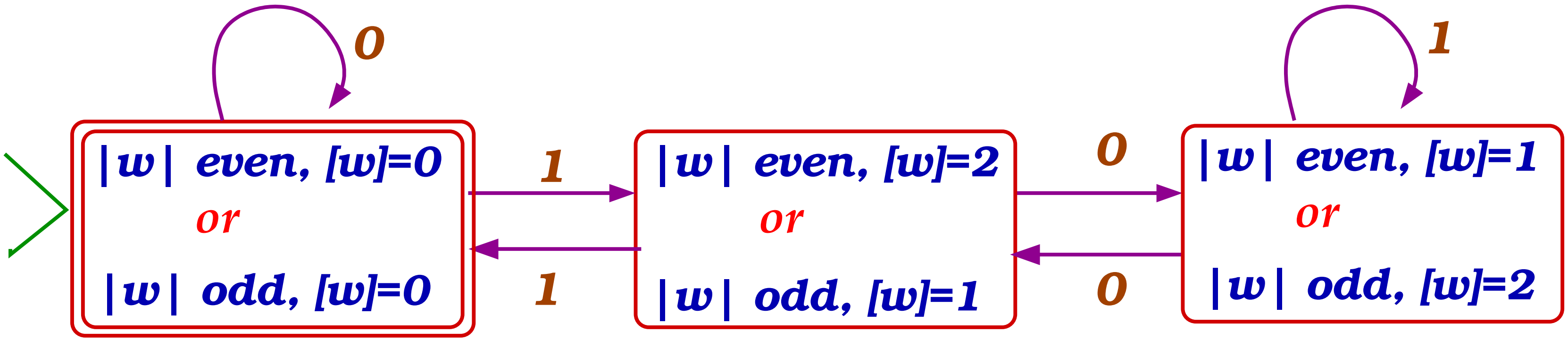
  Either $|w|$ *is even and* $[w] =_3 x$ *or* $|w|$ *is odd and* $[w] =_3 y$

  Let's abbreviate this as (x, y) .

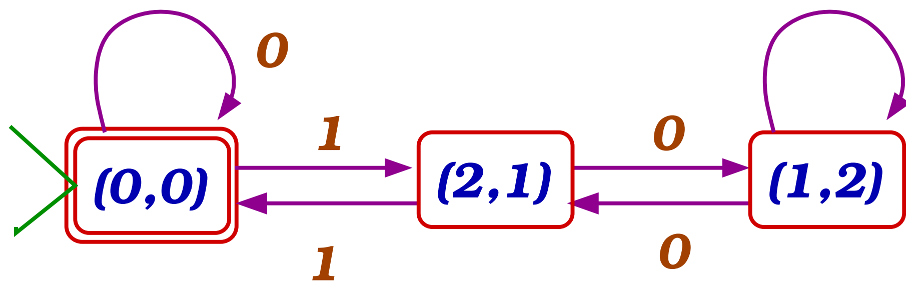
Example: Binary numerals divisible by 3

- For any input w the expectation depends on the parity of $|w|$, the goals are therefore of the form
  Either $|w|$ *is even and* $[w] =_3 x$ *or* $|w|$ *is odd and* $[w] =_3 y$
  Let's abbreviate this as (x, y) .
- From the observation above it follows that $(x,\ y) \xrightarrow{1} (y{+}2,\ x{+}1)$, and $(x,\ y) \xrightarrow{0} (y,\ x)$.

- This yields the following DFA:



Condensed:



RESIDUES AND THEIR APPLICATIONS

More examples of residues

- Take $L =$ English words.

  $L/\mathtt{invent}$ contains the strings $\mathtt{or}, \mathtt{ion}, \mathtt{ive}, \mathtt{ed}$ and $\mathtt{ing}$
  since $\mathtt{inventor}, \mathtt{invention}, \mathtt{inventive}$ and $\mathtt{invented}$ are words.
- ϵ is also in $L/\mathtt{invent}$ since invent is a word.
- The residue $L/\mathtt{ad}$ contains the strings $\mathtt{vance}, \mathtt{apt}, \mathtt{opt}, \mathtt{d},$ and ϵ.
- Take $L = \{\mathrm{ab}\}$, a singleton language.
  We have $L/\varepsilon = \{\mathrm{ab}\}, L/\mathrm{a} = \{\mathrm{b}\}$, and $L/\mathrm{ab} = \varepsilon$.

  For any other string w , $L/w = \emptyset$.
- For any language L we have $L/\varepsilon = L$:

  $w \in L$ iff $\varepsilon \in L/w$.

More examples yet

- $L = \{0,\ 00,\ 010\}$

$$
\begin{aligned}
L/\varepsilon &= L \\
L/0 &= \{\varepsilon,\ 0, 10\} \\
L/00 &= \{\varepsilon\} \\
L/01 &= \{0\} \\
L/010 &= \{\varepsilon\} \\
L/w &= \emptyset \text{for any other} w
\end{aligned}
$$

$L/00 = L/010$, so there are five (different) residues.

An example with language union

- $L = \{\mathtt{a}w \mid w \in \Sigma^*\} \cup \{\mathtt{baa}\}$.

$$\begin{array}{rcll} L/\varepsilon & = & L & \\ L/w & = & \Sigma^* & \text{if } w \text{ starts with } \mathtt{a} \\ L/\mathtt{b} & = & \{\mathtt{aa}\} & \\ L/\mathtt{ba} & = & \{\mathtt{a}\} & \\ L/\mathtt{baa} & = & \{\varepsilon\} & \\ L/w & = & \emptyset & \text{for any other } w \end{array}$$

There are 6 residues.

L and Σ^* are infinite languages, the others are finite.

A single-letter language

- $\Sigma = \{0, 1\}$, $L = \{0\}^*$.

- If $w \in \Sigma^*$ contains 1 then $L/w = \emptyset$.
  Otherwise $L/w = L$.
  There are two residues.

A language based on occurrence count

- $L = \{w \in \{0,1\} \mid \#_0(w)$ **is even** $\}$.

  If $\#_0(w)$ is even then L/w is L,
  otherwise $L/w = \{w \mid \#_0(w)$ is odd $\}$

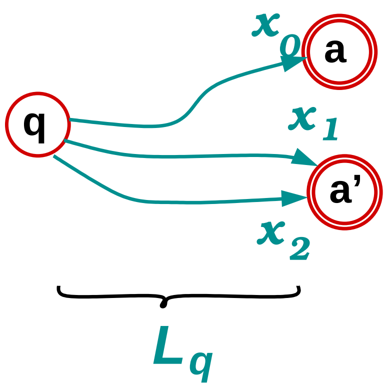
Each state determines a language

- Consider a DFA M recognizing L and a state q in it. Some string x may lead from q to acceptance.

Each state determines a language

- Consider a DFA M recognizing L and a state q in it.
  Some string x may lead from q to acceptance.

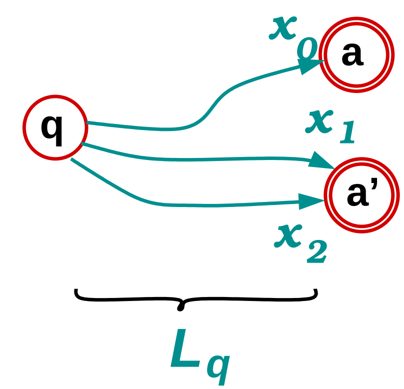
- Denote the set of all such x's by L_q.
  In particular, $L_s = L$.



Each state determines a language

- Consider a DFA M recognizing L and a state q in it.
  Some string x may lead from q to acceptance.

- Denote the set of all such x's by L_q.
  In particular, $L_s = L$.



- Note: We focus on the future of q, not its past!
  (The past would be the set of strings leading to q)

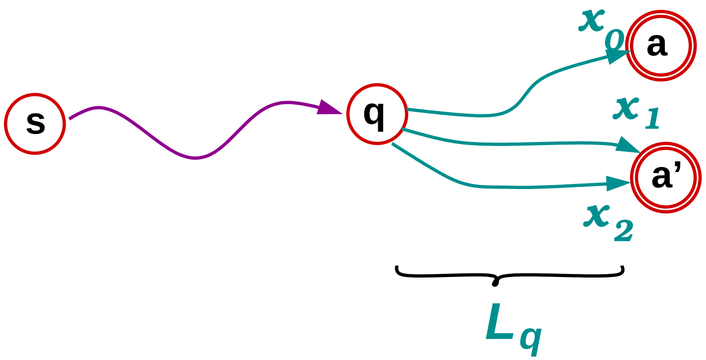
States and residues

- Now suppose that $s \xrightarrow{w} q$.

  A string $w \cdot x$ is accepted by M iff $x \in L_q$.

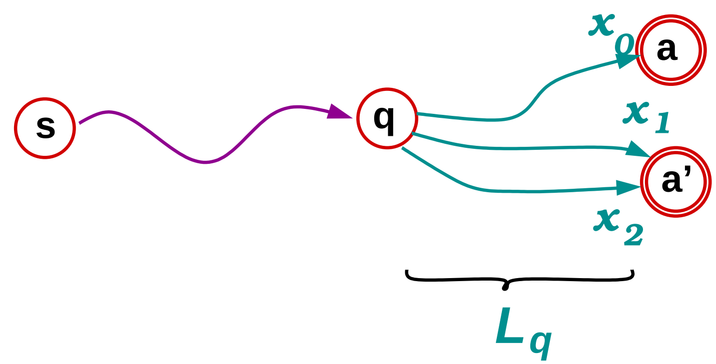
States and residues

- Now suppose that $s \xrightarrow{w} q$.
  A string $w \cdot x$ is accepted by M iff $x \in L_q$.
- x completes w to a string in L :

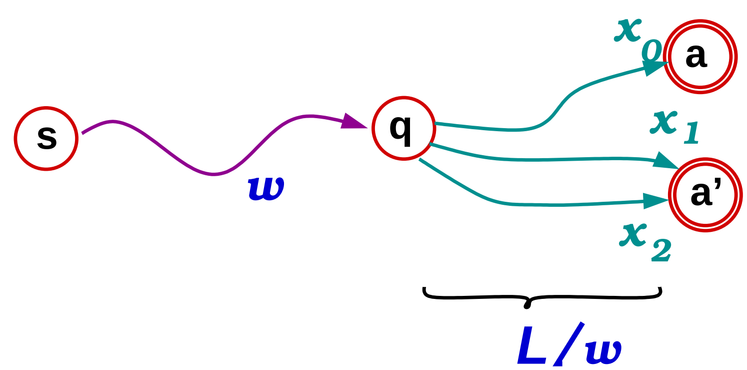


States and residues

- Now suppose that $s \xrightarrow{w} q$.
  A string $w \cdot x$ is accepted by M iff $x \in L_q$.
- x completes w to a string in L:



- L_q is $L/w =$ the residue of L over w:



A property of recognized languages

- **Theorem.** (Myhill-Nerode) *A language recognized by a k-state DFA has $\leqslant k$ residues.*

A property of recognized languages

- **Theorem.** (Myhill-Nerode) *A language recognized by a k-state DFA has $\leqslant k$ residues.*
- Proof. If $s \xrightarrow{u} q$ and $s \xrightarrow{v} q$ then $L/u = L/v$.

A property of recognized languages

- **Theorem.** (Myhill-Nerode) *A language recognized by a* k*-state DFA has* $\leqslant k$ *residues.*
- Proof. If $s \xrightarrow{u} q$ and $s \xrightarrow{v} q$ then $L/u = L/v$.
- Consequently:

  Theorem.

  A language with infinitely many residues is not recognized.

Languages with infinitely many residues

- Let $L = \{w \in \{0, 1\}^* \mid \#_0(w) = \#_1(w)\}$.

Languages with infinitely many residues

- Let $L = \{w \in \{0,1\}^* \mid \#_0(w) = \#_1(w)\}$.
- Consider the residues of L the form $L/1^n \quad (n \geqslant 0)$.

Languages with infinitely many residues

- Let $L = \{w \in \{0,1\}^* \mid \#_0(w) = \#_1(w)\}$.
- Consider the residues of L the form $L/1^n \quad (n \geqslant 0)$.
- For each n we have

$$L/1^n = \{x \mid \#_0(x) = \#_1(x) + n\} ,$$

since to compensate for an initial substring of n 1's the rest of the string should have n extra 0's.

Languages with infinitely many residues

- Let $L = \{w \in \{0,1\}^* \mid \#_0(w) = \#_1(w)\}$.
- Consider the residues of L the form $L/1^n \quad (n \geqslant 0)$.
- For each n we have

  $$L/1^n = \{x \mid \#_0(x) = \#_1(x) + n\},$$

  since to compensate for an initial substring of n 1's the rest of the string should have n extra 0's.
- If $i \neq j$ then $0^i \in L/1^i$ but $\notin L/1^j$ so the two residues are ***different***.

Languages with infinitely many residues

- Let $L = \{w \in \{0,1\}^* \mid \#_0(w) = \#_1(w)\}$.
- Consider the residues of L the form $L/1^n \quad (n \geqslant 0)$.
- For each n we have

  $$L/1^n = \{x \mid \#_0(x) = \#_1(x) + n\} \text{ ,}$$

  since to compensate for an initial substring of n 1's the rest of the string should have n extra 0's.
- If $i \neq j$ then $0^i \in L/1^i$ but $\notin L/1^j$ so the two residues are ***different***.

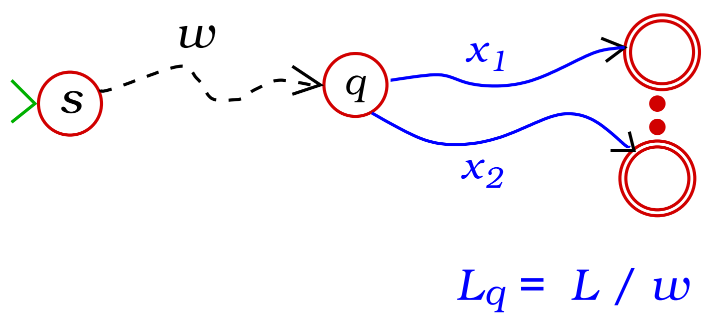
$\therefore$ L is not recognized, since it has infinitely many residues.

States and residues

- We developed automata by thinking of residues as states.
- Let M be an automaton over Σ.
  For a state q of M define

$$L_q =_{\text{df}} \{x \in \Sigma^* \mid q \xrightarrow{x} A\}$$

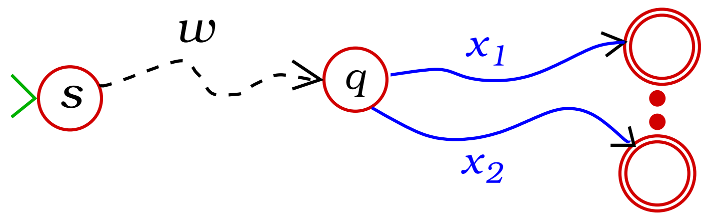
- In particular, for the start state $L_s = L$.
- If $s \xrightarrow{w} q$ then $L_q = L/w$.



⋆ Each string leads from s to some state.

⋆ All strings leading from s to a state q have the same residue.

The Myhill-Nerode Theorem



$L_q = L \,/\, w$

- Every residue L/w is L_q for q as above.
- And two different residues $L/w \neq L/x$ must correspond to two different states.
- So we have an injection that maps residues to states, I.e. the number of residues is bounded by the number of states.
- **Theorem.** (John Myhill and Anil Nerode (1958)) (simplified and rephrased): $\mathcal{L}(M)$ cannot have more residues than M has states.
- Consequence: *A language with infinitely many residues cannot be recognized by any automaton!*

Showing that a language fails recognition

- We saw that $L = \{w \in \{0,1\}^* \mid \#_0(w) = \#_1(w)\}$ has infinitely many residues.
- Consequence: It cannot be recognized by any automaton!!!
- General method: show that L is not recognized
  by showing that there are infinitely many residues.
- We do not need to consider all residues,
  only some infinite selection, defined by a template
- We ***do not need to calculate*** the residues we choose,
  only show that each two of them are different.
- We show them different by exhibiting a string which is in one
  but not in the other.

Example: Unary addition

- Representing unary addition, using unary numerals and the symbols for addition and equality:
- $L = \{\mathtt{1}^k + \mathtt{1}^m = \mathtt{1}^{k+m} \mid k, m \geqslant 1\}$
- What residues would you select?

- $L/\,\mathtt{1}^n + \mathtt{1} =$ for each $n \geqslant 1$.
- Suppose $i \neq j$.
  What string is in $L/\,\mathtt{1}^i + \mathtt{1} =$ but not in $L/\,\mathtt{1}^j + \mathtt{1} =$?

Example: Residues for Mahimahi

- Consider $L = \{u \cdot u \mid u \in \{0,1\}^* \}$.
  What residues L/w to take?

Example: Residues for Mahimahi

- Consider $L = \{u \cdot u \mid u \in \{0,1\}^* \}$.
  What residues L/w to take?

- w with an end-mark would help with differentiating residues.
  Say $0^n 1$?

Example: Residues for Mahimahi

- Consider $L = \{u \cdot u \mid u \in \{0,1\}^* \}$.
  What residues L/w to take?
- w with an end-mark would help with differentiating residues.
  Say $0^n 1$?
- Then $0^i 1 \in L/0^i 1$,
  but for $j > i$ we have $0^i 1 \notin L/0^j 1$,
  because it has two 1's in its first half and none in the second.

Example: Residues for Mahimahi

- Consider $L = \{u \cdot u \mid u \in \{0,1\}^*\}$.
  What residues L/w to take?
- w with an end-mark would help with differentiating residues.
  Say $0^n 1$?
- Then $0^i 1 \in L/0^i 1$,
  but for $j > i$ we have $0^i 1 \notin L/0^j 1$,
  because it has two 1's in its first half and none in the second.
- Since each two of these residues are different,
  L has infinitely many residues,
  and cannot be recognized by a DFA.

Example: Residues for perfect squares

- $L = \{1^{n^2} \mid n \geqslant 0\}$.
- Consider the residues $L/1^{n^2}$ for each $n > 0$.
- The first perfect square following n^2 is $(n+1)^2 = n^2 + 2n + 1$.
- So the shortest non-null string of $L/1^{i^2}$ is 1^{2i+1}.
- It follows that $1^{2i+1} \in L/1^{i^2}$
  but $1^{2i+1} \notin L/1^{j^2}$ for any $j > i$.
- Since every two of these residues are different,
  L has infinitely many residues,
  and cannot be recognized by any automaton.

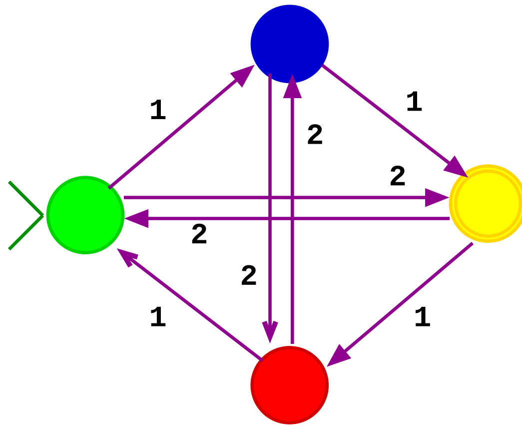
Building automata directly from residues

- We showed that every recognized language has finitely many residues.
- The converse is also true:
- If $L \subseteq \Sigma^*$ has finitely many residues, then $L = \mathcal{L}(M)$ where:
  - ⋆ The states of M are the residues.
  - ⋆ The initial state is $L/\varepsilon = L$.
  - ⋆ A state L/w is accepting iff it contains $\boldsymbol{\varepsilon}$.
  - ⋆ The transitions are given by $L/w \xrightarrow{\sigma} L/w\sigma$.
- We used the same idea to construct automata, except that here we assume that the residues are given to us.
- We write ***Res***(L) for the automaton constructed from residues.

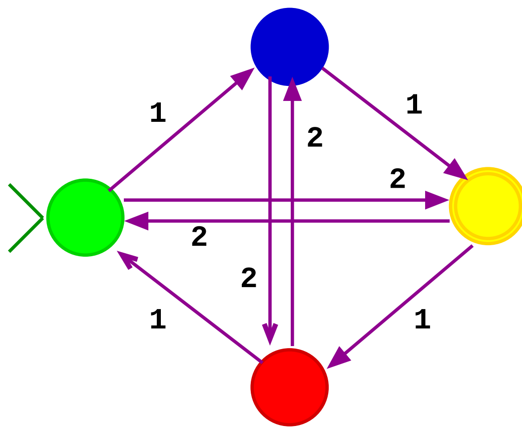
Recognized = finitely many residues

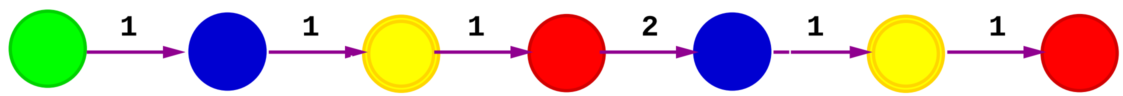
- A language L is recognized iff it has finitely many residues.
- The DFA constructed from L's residues
  has the fewer states
- Given a DFA M recognizing L, and a state q,

AUTOMATA ARE REPETITIVE

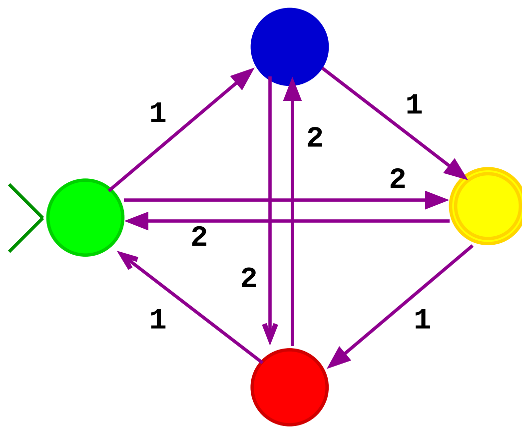


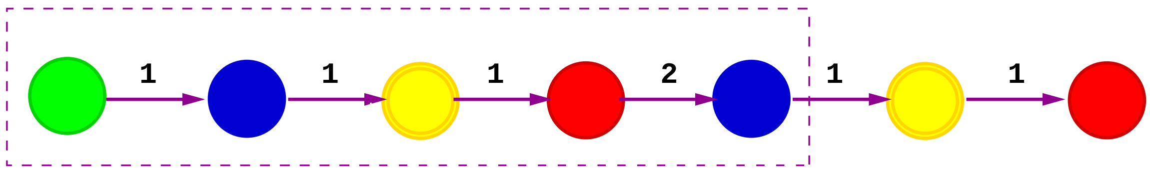
- Here's an automaton that accepts a string $w \in \{1, 2\}^*$ iff the sum of the digits in w is $2 \mod (4)$.



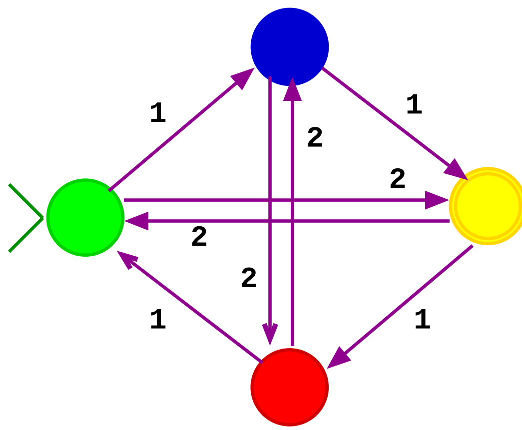


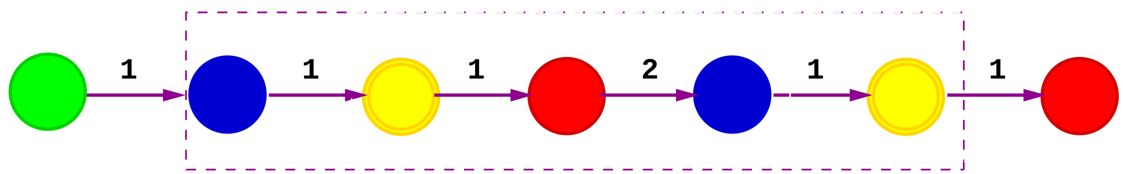
- This is its trace for input 111212.
  The input has 6 symbols, so the trace lists 7 states.



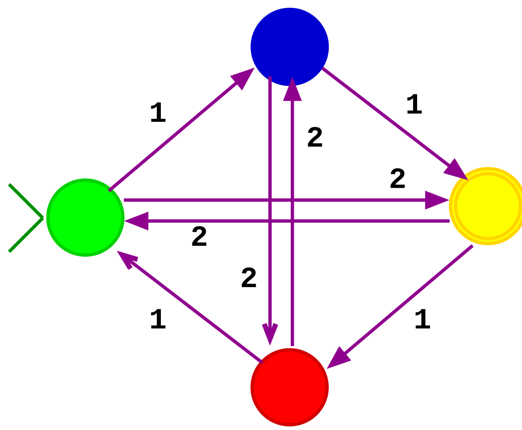


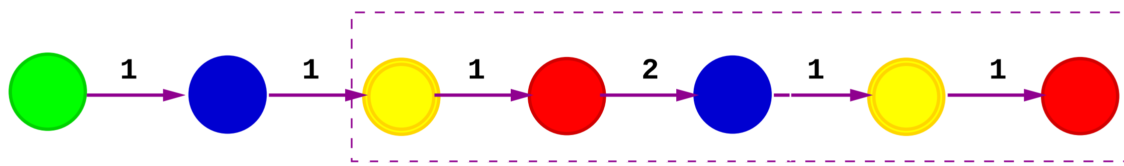
- Looking at the first 5 of the 7, we must have a state repeating, because there are only 4 states.





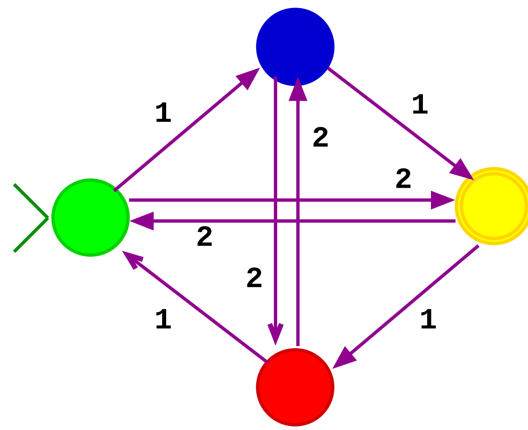
The same happens for the next stretch of 5 states (i.e. 4 input symbols)



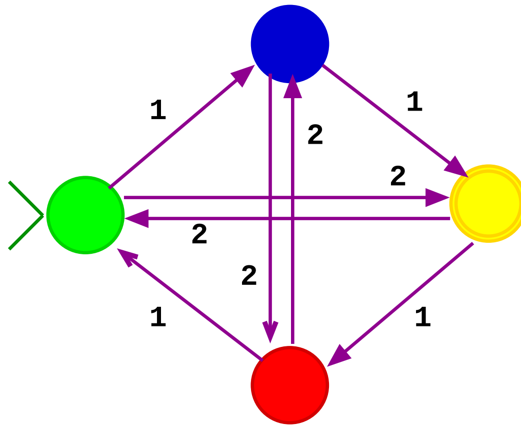


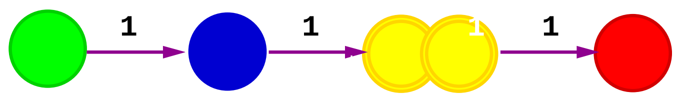
And the next one.

No matter which window of 5 states we take there will be a state repeating!



We can short-cut the steps from the yellow state to itself,
and the result will still be a legit trace, but for 112.





We can short-cut the steps from the yellow state to itself,
and the result will still be a legit trace, but for 112.

The Shortcut Theorem

- **Theorem.** *Let* M *be a* k*-state DFA.*
  If $q \xrightarrow{u} p$ *and* $|u| \geqslant k$ *then*
  $q \xrightarrow{u'} p$ *where* u' *is* u *with some substring* $y \neq \varepsilon$ *clipped off, i.e. removed.*

The Shortcut Theorem

- **Theorem.** *Let* M *be a* k *-state DFA.*
  If $q \xrightarrow{u} p$ *and* $|u| \geqslant k$ *then*
  $q \xrightarrow{u'} p$ *where* u' *is* u *with some*
  substring $y \neq \varepsilon$ *clipped off, i.e. removed.*
- Suppose we have $s \xrightarrow{w_0} p \xrightarrow{u} q \xrightarrow{w_1} A$ with $|u| \geqslant k$.

The Shortcut Theorem

- **Theorem.** *Let* M *be a* k *-state DFA.*
  If $q \xrightarrow{u} p$ *and* $|u| \geqslant k$ *then*
  $q \xrightarrow{u'} p$ *where* u' *is* u *with some*
  substring $y \neq \varepsilon$ *clipped off, i.e. removed.*
- Suppose we have $s \xrightarrow{w_0} p \xrightarrow{u} q \xrightarrow{w_1} A$ with $|u| \geqslant k$.

  Then $s \xrightarrow{w_0} p \xrightarrow{u'} q \xrightarrow{w_1} A$

The Clipping Theorem

- **Theorem.** *If a k-state DFA accepts a string w,*
  and u is a substring of w of length $\geqslant k$,
  then u has a substring $y \neq \varepsilon$ such that
  w with y removed is also accepted.

The Clipping Theorem

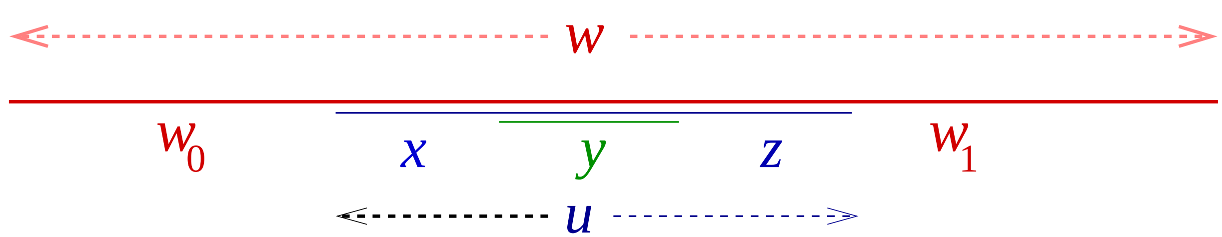
- **Theorem.** *If a k -state DFA accepts a string w , and u is a substring of w of length $\geqslant k$, then u has a substring $y \neq \varepsilon$ such that w with y removed is also accepted.*
- That is, if M accepts $w_0 \cdot u \cdot w_1$, where $|u| \geqslant k$, then there is a split $u = x \cdot y \cdot z$, with $y \neq \varepsilon$, such that $w' = w_0 \cdot x \cdot z \cdot w_1$ is also accepted.

The Clipping Theorem

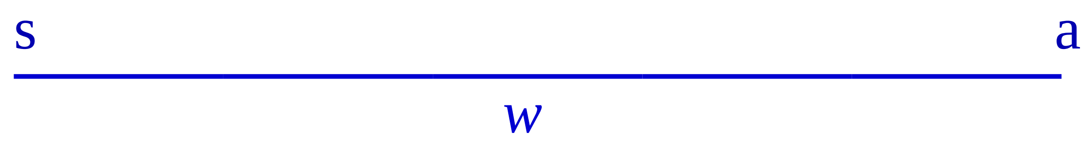
- **Theorem.** *If a k-state DFA accepts a string w, and u is a substring of w of length $\geqslant k$, then u has a substring $y \neq \varepsilon$ such that w with y removed is also accepted.*
- That is, if M accepts $w_0 \cdot u \cdot w_1$, where $|u| \geqslant k$, then there is a split $u = x \cdot y \cdot z$, with $y \neq \varepsilon$, such that $w' = w_0 \cdot x \cdot z \cdot w_1$ is also accepted.
- We call u the ***critical*** substring, the occurrence of y the ***clipped*** substring, and w' the ***reduced*** string.

The Clipping Theorem

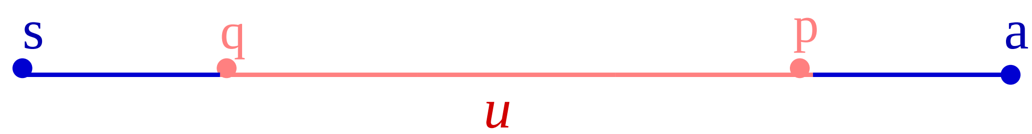
- **Theorem.** *If a k-state DFA accepts a string w, and u is a substring of w of length $\geqslant k$, then u has a substring $y \neq \varepsilon$ such that w with y removed is also accepted.*
- We call u the ***critical*** substring, the occurrence of y the ***clipped*** substring, and w' the ***reduced*** string.



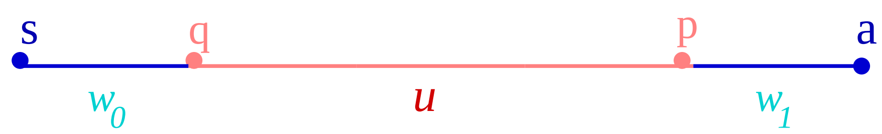
Clipping step by step



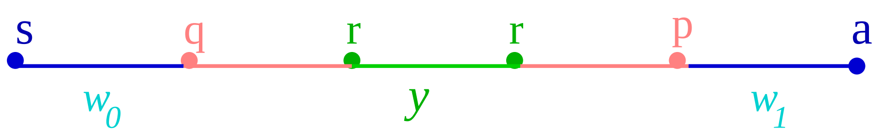
Clipping step by step



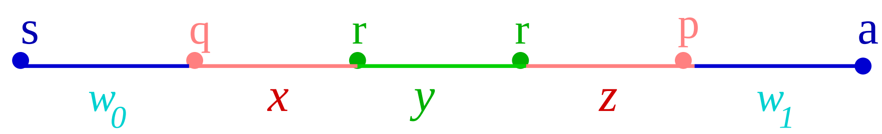
Clipping step by step



Clipping step by step

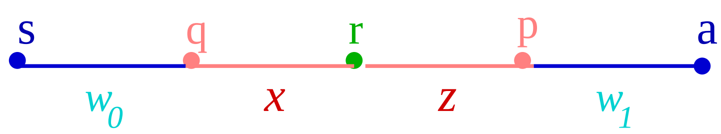


Clipping step by step



Clipping step by step

Clipping step by step



An application: the shortest string accepted

- If M is a 10 state automaton that accepts some string. What is the length ℓ of the ***shortest*** string accepted?
  1. $\ell \in [30..100]$
  2. $\ell \in [10..25]$
  3. $\ell \in [0..9]$
  4. Can't tell, could be anything.

An application: the shortest string accepted

- If M is a 10 state automaton that accepts some string. What is the length ℓ of the ***shortest*** string accepted?
- **Theorem.** *If a k-state automaton M accepts some string, then it accepts a string of length $< k$.*

An application: the shortest string accepted

- If M is a 10 state automaton that accepts some string. What is the length ℓ of the ***shortest*** string accepted?

- **Theorem.** *If a k-state automaton M accepts some string, then it accepts a string of length $< k$.*

- **Proof:** Let w be a shortest string accepted by M.
  If $|w| \geqslant k$ then we invoke the Clipping Theorem,
  with w itself for u,
  and obtain a $w' \in L$ shorter than w.
  This contradicts the assumed minimality of $|w|$.

On not being an insect

- How do you tell that the critter on your desk is not an insect?

On not being an insect

- How do you tell that the critter on your desk is not an insect?
- Check that it violates some property of insects, e.g. it has eight rather than six legs.
- How do you tell that a given language L is not recognized by any automaton?
- Refer to a property that all recognized languages have, but L does not.

On not being an insect

- How do you tell that the critter on your desk
  is not an insect?
- Check that it violates some property of insects,
  e.g. it has eight rather than six legs.
- How do you tell that a given language L
  is not recognized by any automaton?
- Refer to a property that all recognized languages have,
  but L does not.

The Clipping Property

- The Clipping Theorem:

  Every recogized L has this ***Clipping Property:***

The Clipping Property

- The Clipping Theorem:
  Every recogized L has this **_Clipping Property:_**
  - There is a k (# of states of an acceptor for L),
  - so that for every $w \in L$ and substring u of length $\geqslant k$,
  - u has a “clippable” substring $y \neq \varepsilon$:
    removing y from w yields a string in L.

The Clipping Property

- The Clipping Theorem:
  Every recogized L has this **Clipping Property:**
  - There is a k (# of states of an acceptor for L),
  - so that for every $w \in L$ and substring u of length $\geqslant k$,
  - u has a “clippable” substring $y \neq \varepsilon$:
    removing y from w yields a string in L .
- A language ***fails Clipping*** when
  - for any $k > 0$
  - we can choose $w \in L$ and substring u of length $\geqslant k$,
  - so that ***any*** clipping off u yields $w' \notin L$.

Example: an-bn

- Let $L = \{\mathrm{a}^n\mathrm{b}^n \mid n \geqslant 0\}$
- L fails clipping:

1. Let $k > 0$
2. Choose $w = \mathrm{a}^k\mathrm{b}^k$ and $u = \mathrm{a}^k$.
   We have $w \in L$ and $|u| \geqslant k$.
3. Any clipping in u yields from w
   a w' of the form $\mathrm{a}^p\mathrm{b}^k$ with $p < k$.
   So $w' \notin L$.

- Consequence: L fails the Clipping Property and cannot be recognized.

Example: Unary addition

- Consider the strings representing addition in unary:
  $A = \{1^p + 1^q = 1^{p+q} \mid p, q > 0\}$.
- A fails the Clipping Property:

  1. Let $k > 0$.
  2. Choose $w = 1^k + 1 = 1^{k+1}$
     and u the substring 1^{k+1}.
     $w \in A$ and $|u| \geqslant k$.
  3. Any clipping in u yields from w a string
     $w' = 1^\ell + 1 = 1^{k+1}$ with $\ell < k$.
     $w' \notin A$.

- A fails Clipping, and so cannot be recognized.

Example: Perfect squares in unary

- Consider $L = \{1^{n^2} \mid n \geqslant 0\}$.
- L fails the Clipping Property:

1. Let $k > 0$.
2. Choose $w = 1^{k^2}$ and $u = 1^k$.
   $w \in L$ and $|u| \geqslant k$.
3. For any clipped y we have $1 \leqslant |y| \leqslant |u| = k$,
   so for the reduced string $w' = 1^\ell$ where $k^2 - k \leqslant \ell < k^2$.
   $w' \notin L$ because ℓ cannot be a square: the largest square preceding k^2 is $(k-1)^2 = k^2 - 2k + 1$ which is $< k^2 - k \leqslant \ell$.

- So L fails Clipping, and cannot be recognized.

Example: The mahimahi language

- Consider $L = \{x \cdot x \mid x \in \{0, 1\}^*\}$
- Idea: Take $w = x \cdot x$ with x that starts with a marker.

Example: The mahimahi language

- Consider $L = \{x \cdot x \mid x \in \{0, 1\}^*\}$
- Idea: Take $w = x \cdot x$ with x that starts with a marker.

1. Let $k > 0$.
2. Choose $w = 01^k01^k$ and $u =$ left substring 1^k in w.
   $w \in L$ and $|u| \geqslant k$.

Example: The mahimahi language

- Consider $L = \{x \cdot x \mid x \in \{0,1\}^*\}$
- Idea: Take $w = x \cdot x$ with x that starts with a marker.

1. Let $k > 0$.
2. Choose $w = 01^k01^k$ and $u =$ left substring 1^k in w.
   $w \in L$ and $|u| \geqslant k$.
3. Any clipped y in u yields from w
   a reduced string $w' = 01^\ell 01^k$
   where $\ell < k$.
   Such w' cannot be of the form xx,
   because its first half starts with 0
   while its second half starts with 1.

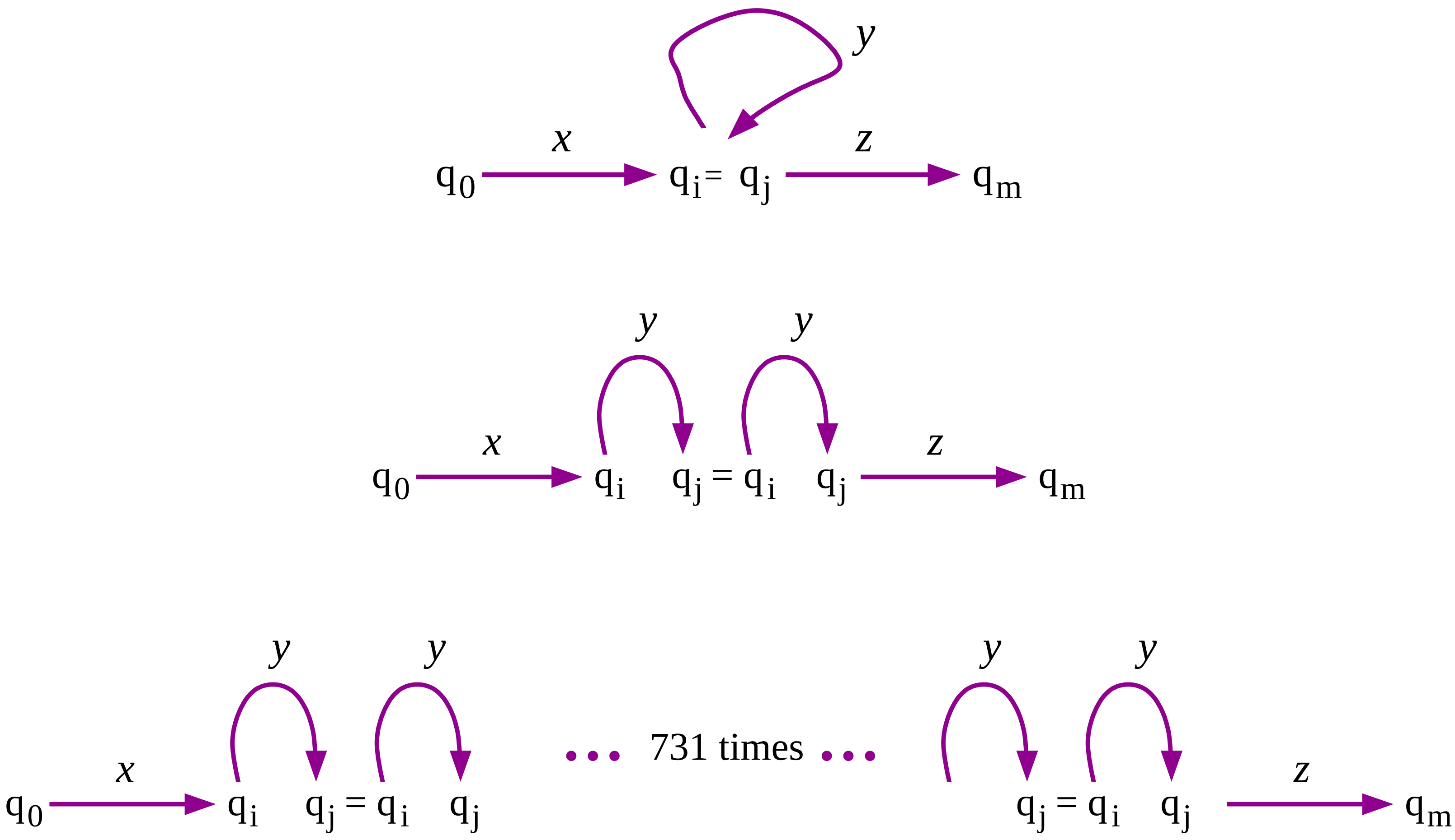
Example: The mahimahi language

- Consider $L = \{x \cdot x \mid x \in \{0, 1\}^*\}$
- Idea: Take $w = x \cdot x$ with x that starts with a marker.

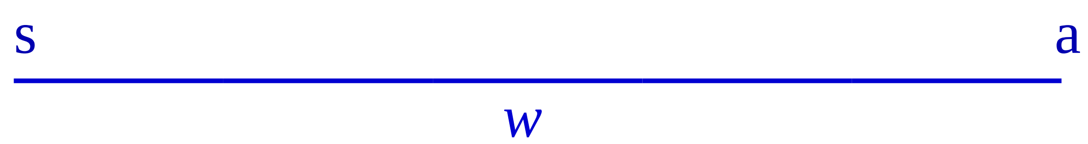
1. Let $k > 0$.
2. Choose $w = 01^k01^k$ and $u =$ left substring 1^k in w.
   $w \in L$ and $|u| \geqslant k$.
3. Any clipped y in u yields from w
   a reduced string $w' = 01^\ell 01^k$
   where $\ell < k$.
   Such w' cannot be of the form xx,
   because its first half starts with 0
   while its second half starts with 1.

- L fails the Clipping Property, and cannot be recognized.

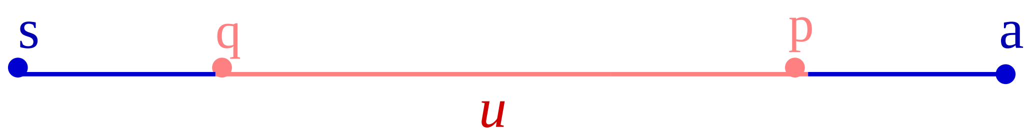
Pumping up rather than clipping



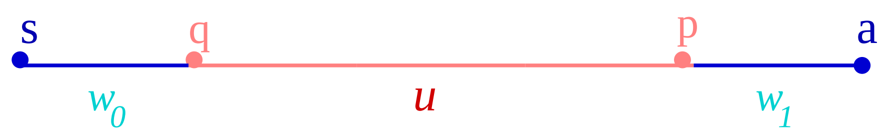
Pumping step-by-step



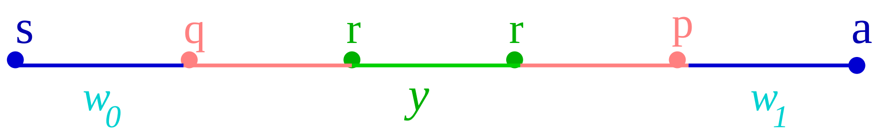
Pumping step-by-step



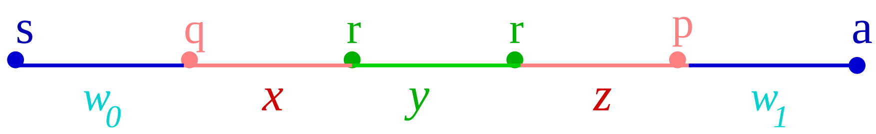
Pumping step-by-step



Pumping step-by-step

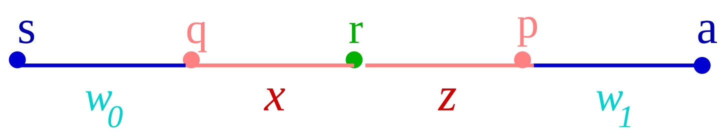


Pumping step-by-step

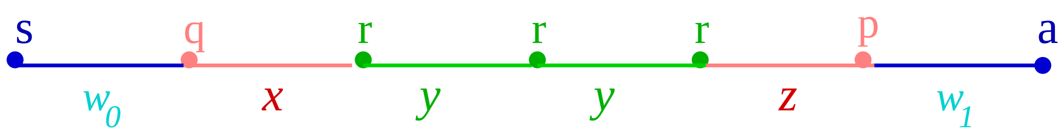


Pumping step-by-step

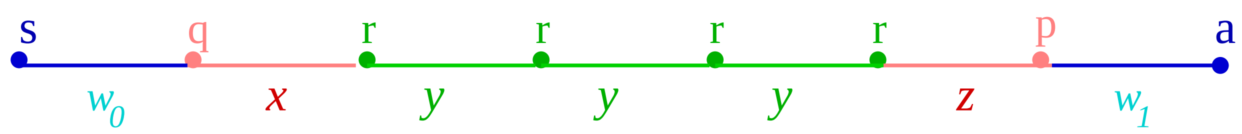
Pumping step-by-step



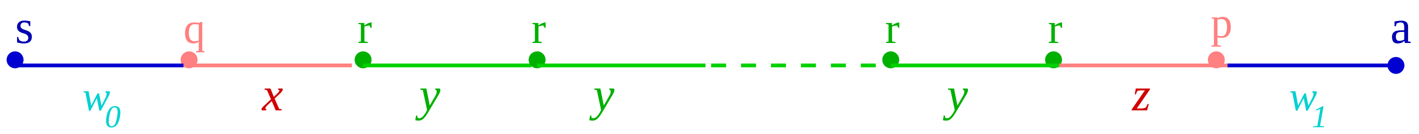
Pumping step-by-step



Pumping step-by-step



Pumping step-by-step



Pumping instances

- Let $w \in \Sigma^*$ and
  y a particular substring of w : $w = x \cdot y \cdot z$.
- The ***n-th pumping instance*** of $w = x \cdot y \cdot z$
  over (the exhibited occurrence of) y
  is defined to be $x \cdot y^n \cdot z$.

The Pumping Theorem

- Let M be a k-state DFA over Σ, $L = \mathcal{L}(M)$.
- As for Clipping, choose $w \in L$ and a substring u of w of length $\geqslant k$.
- CONCLUDE: u has a non-empty substring y
  such that all pumping instances of w over y are in L.
- Recall: The n*-th pumping instance* of w over
  (a particular occurrence of) y
  is the result of replacing y by y^n.

Failing Pumping

A language ***fails Pumping*** when:

1. For any $k > 0$
2. there are $w \in L$
   and substring u of w of length $\geqslant k$
3. so that for ***every*** y within u
   there is a pumping instance w over y which is not in L.

Example: The Primes

- $L = \{1^p \mid p \text{ is prime }\}$
- Suppose L is recognized by a k-state DFA M.

Example: The Primes

- $L = \{1^p \mid p \text{ is prime }\}$
- Suppose L is recognized by a k-state DFA M.
- Take a prime $p > k$ and $w = 1^p \in L$.
- There is a pumping segment y in w of length $\ell \neq 0$.

Example: The Primes

- $L = \{1^p \mid p \text{ is prime }\}$
- Suppose L is recognized by a k-state DFA M.
- Take a prime $p > k$ and $w = 1^p \in L$.
- There is a pumping segment y in w of length $\ell \neq 0$.
- The $(p+1)$-st pumping instance of w over y
  has length $|w| - \ell + (p+1)\ell = p + p\ell = p(\ell + 1)$,
  which is not prime.

Example: The Primes

- $L = \{1^p \mid p \text{ is prime }\}$
- Suppose L is recognized by a k-state DFA M.
- Take a prime $p > k$ and $w = 1^p \in L$.
- There is a pumping segment y in w of length $\ell \neq 0$.
- The $(p+1)$-st pumping instance of w over y has length $|w| - \ell + (p+1)\ell = p + p\ell = p(\ell + 1)$, which is not prime.
- Contradiction. M cannot exist.

Example: Necessary use of Pumping

- Show that the language

$$L = \{ w \cdot \mathtt{a}^n \mid w \in \{\mathtt{a}, \mathtt{b}\}^*,\ \#_a(w) = n \}$$

  is not recognized.

Example: Necessary use of Pumping

- Show that the language

$$L = \{w \cdot \mathtt{a}^n \mid w \in \{\mathtt{a}, \mathtt{b}\}^*,\ \#_a(w) = n\ \}$$

  is not recognized.

- Suppose L were recognized by a k-state DFA.
  Let $w = \mathtt{b}^k\mathtt{a}^k$, which is in L,
  and take $u = \mathtt{b}^k$, the prefix of w.

Example: Necessary use of Pumping

- Show that the language
$$L = \{w \cdot \mathtt{a}^n \mid w \in \{\mathtt{a},\mathtt{b}\}^*,\ \#_a(w) = n\ \}$$
is not recognized.

- Suppose L were recognized by a k-state DFA.
Let $w = \mathtt{b}^k\mathtt{a}^k$, which is in L,
and take $u = \mathtt{b}^k$, the prefix of w.

- By the Pumping Theorem u has a substring $y = \mathtt{b}^\ell$ where $\ell > 0$ such that $\mathtt{b}^{k+n\ell}\,\mathtt{a}^k \in L$ for all $n \geqslant 0$. In particular, for $n = 1$ we have $w' = \mathtt{b}^{k+\ell}\,\mathtt{a}^k \in L$.

Example: Necessary use of Pumping

- Show that the language
$$L = \{w \cdot \mathtt{a}^n \mid w \in \{\mathtt{a}, \mathtt{b}\}^*,\ \#_a(w) = n\ \}$$
is not recognized.

- Suppose L were recognized by a k-state DFA.
Let $w = \mathtt{b}^k \mathtt{a}^k$, which is in L,
and take $u = \mathtt{b}^k$, the prefix of w.

- By the Pumping Theorem u has a substring $y = \mathtt{b}^\ell$ where $\ell > 0$ such that $\mathtt{b}^{k+n\ell}\mathtt{a}^k \in L$ for all $n \geqslant 0$. In particular, for $n = 1$ we have $w' = \mathtt{b}^{k+\ell}\mathtt{a}^k \in L$.
But this is impossible, because the second half of this w'
has $\mathtt{b}$'s, so $w' \notin L$.

- Thus no DFA recgnizes L .

Minimum states for finite language recognition

- Any ***finite*** language L is recognized by an automaton!
- But how many states are needed?

Minimum states for finite language recognition

- Any ***finite*** language L is recognized by an automaton!
- But how many states are needed?
- At least as many as the longest string-length in L.

Minimum states for finite language recognition

- Any ***finite*** language L is recognized by an automaton!
- But how many states are needed?
- At least as many as the longest string-length in L.
- Proof: If M with k states recognizes a string longer than k, then Pumping applies, and L is infinite!

MODIFYING & COMBINING AUTOMATA

Partial-automata

- A ***partial-automaton*** is an automaton whose transition mapping is a ***partial*** function (recall that a total-function is also a partial-function).

Partial-automata

- A ***partial-automaton*** is an automaton whose transition mapping is a ***partial*** function (recall that a total-function is also a partial-function).
- A partial-automaton M terminates execution when it cannot proceed: no applicable transition (due to partiality) or no next-letter to move to.

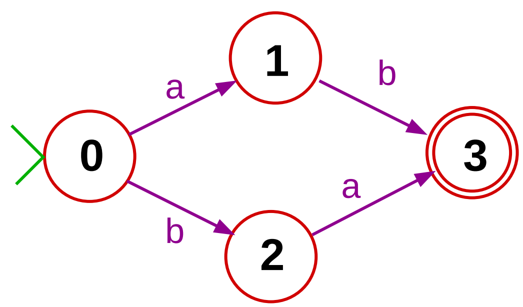
  It ***accepts*** w if its state-trace for w ends with an accepting state.

Partial-automata

- A ***partial-automaton*** is an automaton whose transition mapping is a ***partial*** function (recall that a total-function is also a partial-function).
- A partial-automaton M terminates execution when it cannot proceed: no applicable transition (due to partiality) or no next-letter to move to.

It ***accepts*** w if its state-trace for w ends with an accepting state.

- Example: A partial automaton recognizing $\{\mathtt{ab}, \mathtt{ba}\}$:



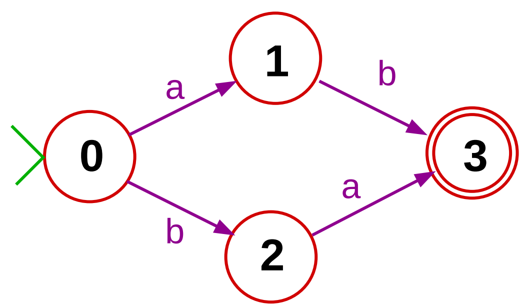
Partial-automata

- A ***partial-automaton*** is an automaton whose transition mapping is a ***partial*** function (recall that a total-function is also a partial-function).

- A partial-automaton M terminates execution when it cannot proceed: no applicable transition (due to partiality) or no next-letter to move to.

  It ***accepts*** w if its state-trace for w ends with an accepting state.

- Example: A partial automaton recognizing $\{\texttt{ab}, \texttt{ba}\}$:



- Some people use "automaton" for our "partial-automaton" and "total-automaton" for our "automaton."

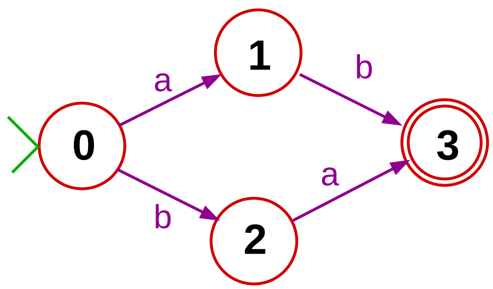
From partial- to total-automaton

- **Theorem.** Every partial-automaton M can be converted into a total-automaton $\bar{M}$ equivalent to M, i.e. recognizing the same language.
  Do you seee how?

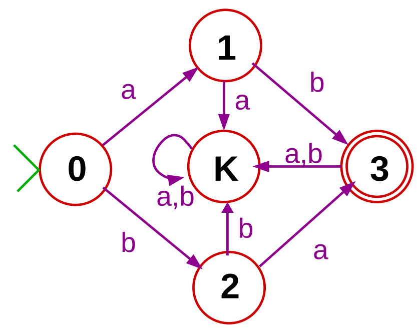
From partial- to total-automaton

- **Theorem.** Every partial-automaton M can be converted into a total-automaton $\bar{M}$ equivalent to M, i.e. recognizing the same language.
  Do you seee how?
- Just add a sink to M :

convert



to

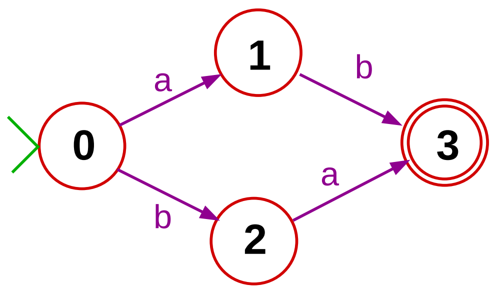


From partial- to total-automaton

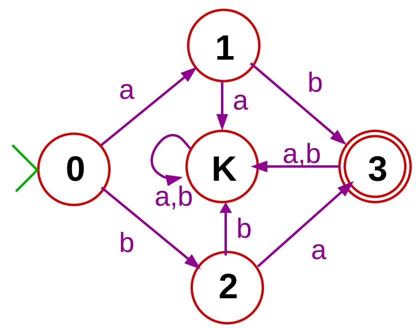
- **Theorem.** Every partial-automaton M can be converted into a total-automaton $\bar{M}$ equivalent to M, i.e. recognizing the same language.
  Do you seee how?

- Just add a sink to M :

convert



to



- That is, $\bar{M}$ is obtained by adding to M
  a sink state K , with all missing transitions of M
  as well as outgoing transition from K , pointing to K .

Application: Additional languages recognized

- Suppose M recognizes $\{w \in \{\mathtt{a}, \mathtt{b}\}^* \mid \#_a(w) = \#_b(w) \mod 2\}$.
- Then swapping states in M yields an automaton recognizing

$$\{w \in \{\mathtt{a}, \mathtt{b}\}^* \mid \#_a(w) \neq \#_b(w) \mod 2\}$$

Application: Showing a language not-recognized

- Show $L = \{w \in \{\mathtt{a}, \mathtt{b}\}^* \mid \#_a(w) \neq \#_b(w)\}$ is not recognized.

  ow observe that $L' = \bar{L} \cap \{\mathtt{a}\}^* \cdot \{\mathtt{b}\}^*$.

Application: Showing a language not-recognized

- Show $L = \{w \in \{\mathtt{a}, \mathtt{b}\}^* \mid \#_a(w) \neq \#_b(w)\}$ is not recognized.
- Clipping doesn't work!

  ow observe that $L' = \bar{L} \cap \{\mathtt{a}\}^* \cdot \{\mathtt{b}\}^*$.

Application: Showing a language not-recognized

- Show $L = \{w \in \{\mathtt{a}, \mathtt{b}\}^* \mid \#_a(w) \neq \#_b(w)\}$ is not recognized.
- Clipping doesn't work!
- Use Clipping to show that

$$L' = \{w \in \{\mathtt{a}, \mathtt{b}\}^* \mid \#_a(w) = \#_b(w)\}$$

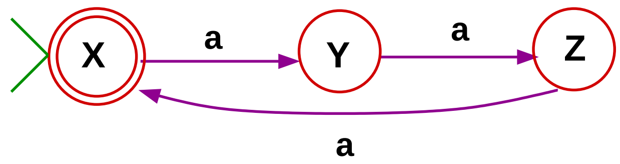
is not recognized.

ow observe that $L' = \bar{L} \cap \{\mathtt{a}\}^* \cdot \{\mathtt{b}\}^*$.

Combining two automata

Let $\Sigma = \{\mathtt{a}, \mathtt{b}\}$.

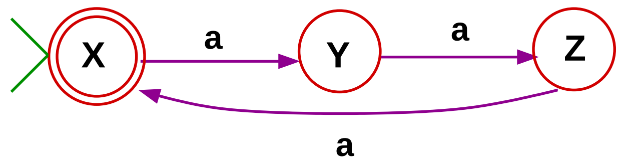
- Suppose M_3 recognizes $L_3 = \{w \in \Sigma^* \mid \#_a(w) = 0 \mod (3)\}$



Combining two automata

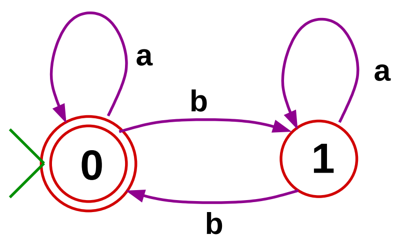
Let $\Sigma = \{\mathtt{a}, \mathtt{b}\}$.

- Suppose M_3 recognizes $L_3 = \{w \in \Sigma^* \mid \#_a(w) = 0 \mod (3)\}$



and

- M_2 recognizes $L_2 = \{w \in \Sigma^* \mid \#_b(w) = 0 \mod (2)\}$.

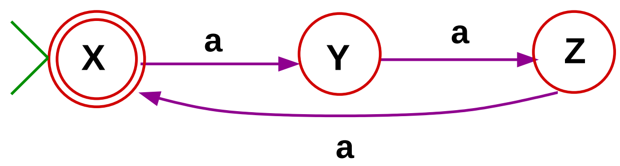


$\#_b w = 0 \mod 2$

Combining two automata

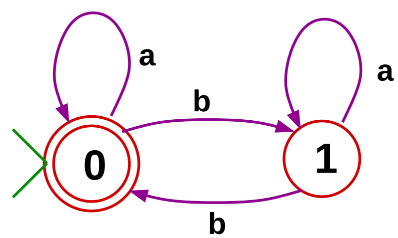
Let $\Sigma = \{\mathtt{a}, \mathtt{b}\}$.

- Suppose M_3 recognizes $L_3 = \{w \in \Sigma^* \mid \#_a(w) = 0 \mod (3)\}$



and

- M_2 recognizes $L_2 = \{w \in \Sigma^* \mid \#_b(w) = 0 \mod (2)\}$.



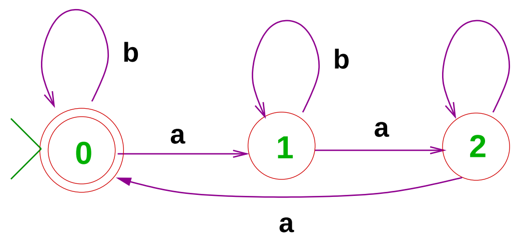
$\#_b w = 0 \mod 2$

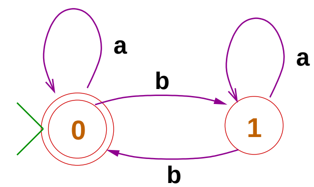
This is special parallelism:

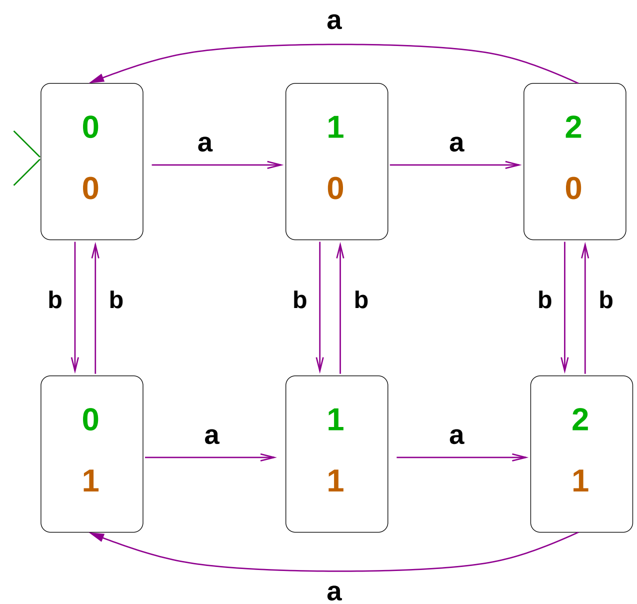
the two processors may work in tandem,

because they read the same input one symbol at a time.

Two automata collaborating

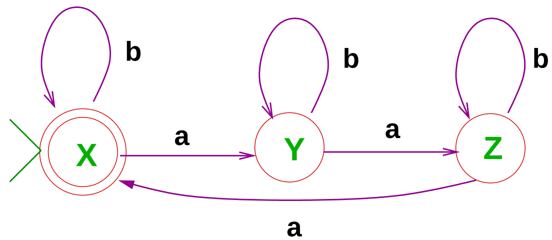


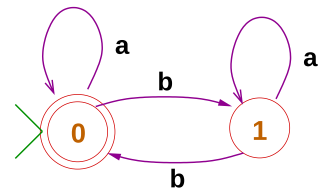


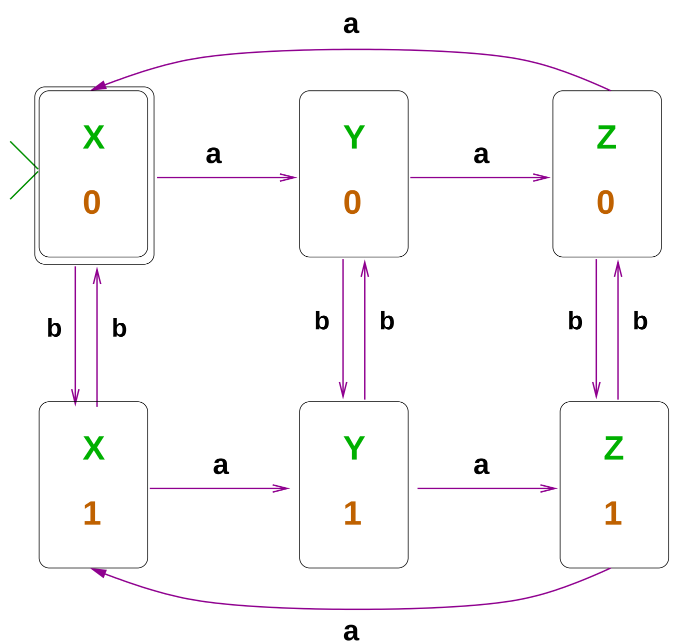


Conjuctive pairing

- Accepting when both accept:



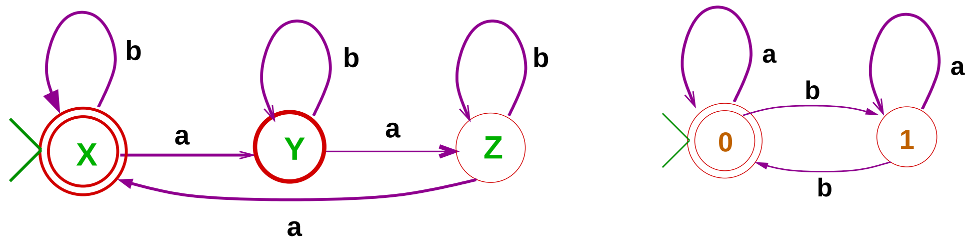


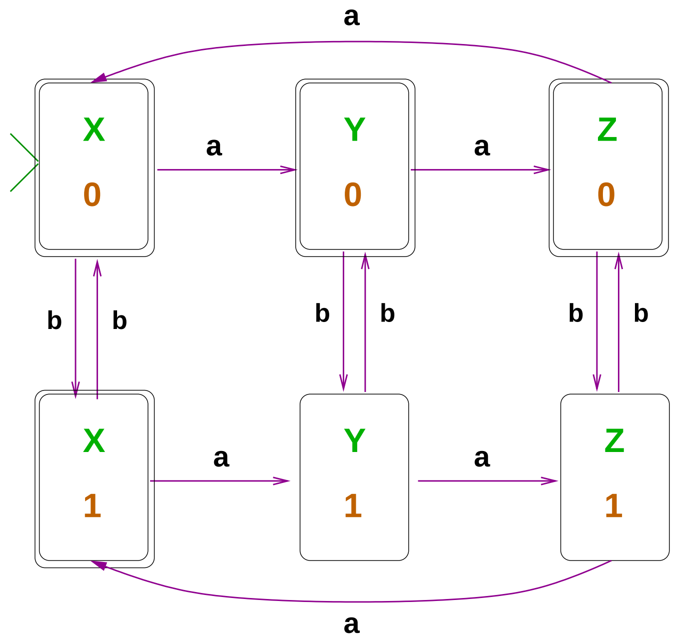


both accept

Disjunctive pairing

- Accepting when at least one automaton accepts:





at least one accepts

Formal definition of automata product

- Given automata $M = (\Sigma, Q, s, A, \delta)$ and $M' = (\Sigma, Q', s', A', \delta')$ consider a ***coupling***:

Formal definition of automata product

- Given automata $M = (\Sigma, Q, s, A, \delta)$ and $M' = (\Sigma, Q', s', A', \delta')$ consider a ***coupling***:
  - States are pairs $\langle q, q' \rangle$ where $q \in Q$ and $q' \in Q'$. I.e. the set of states is $Q \times Q'$.
  - The initial state is $\langle s, s' \rangle$.

Formal definition of automata product

- Given automata $M = (\Sigma, Q, s, A, \delta)$ and $M' = (\Sigma, Q', s', A', \delta')$ consider a ***coupling***:
  - States are pairs $\langle q, q' \rangle$ where $q \in Q$ and $q' \in Q'$.
    I.e. the set of states is $Q \times Q'$.
  - The initial state is $\langle s, s' \rangle$.
  - The transitions are $\langle q, q' \rangle \xrightarrow{\sigma} \langle p, p' \rangle$ where $q \xrightarrow{\sigma} p$ in M and $q' \xrightarrow{\sigma} p'$ in M'.

Formal definition of automata product

- Given automata $M = (\Sigma, Q, s, A, \delta)$ and $M' = (\Sigma, Q', s', A', \delta')$ consider a ***coupling***:
  - States are pairs $\langle q, q' \rangle$ where $q \in Q$ and $q' \in Q'$. I.e. the set of states is $Q \times Q'$.
  - The initial state is $\langle s, s' \rangle$.
  - The transitions are $\langle q, q' \rangle \xrightarrow{\sigma} \langle p, p' \rangle$ where $q \xrightarrow{\sigma} p$ in M and $q' \xrightarrow{\sigma} p'$ in M'.
- In a ***conjunctive product*** the set of accepting states is $A \times A'$ (both automata accept).

Formal definition of automata product

- Given automata $M = (\Sigma, Q, s, A, \delta)$ and $M' = (\Sigma, Q', s', A', \delta')$ consider a ***coupling***:
  - States are pairs $\langle q, q' \rangle$ where $q \in Q$ and $q' \in Q'$. I.e. the set of states is $Q \times Q'$.
  - The initial state is $\langle s, s' \rangle$.
  - The transitions are $\langle q, q' \rangle \xrightarrow{\sigma} \langle p, p' \rangle$ where $q \xrightarrow{\sigma} p$ in M and $q' \xrightarrow{\sigma} p'$ in M'.
- In a ***conjunctive product*** the set of accepting states is $A \times A'$ (both automata accept).
- In a ***disjunctive product*** the set of accepting states is $(A \times Q') \cup (Q \times A')$ ($\geqslant 1$ accept).

Some applications

- $L = \{\ \mathtt{a}\, w\, \mathtt{z}\ \mid\ w \in \Sigma^*\ \}$

Some applications

- $L = \{ \texttt{a}\, w\, \texttt{z} \mid w \in \Sigma^* \}$
- $\{\texttt{a}^p\texttt{b}^q \mid p \text{ is odd } \}$.

Some applications

- $L = \{ \, \mathtt{a} \, w \, \mathtt{z} \;\mid\; w \in \Sigma^* \, \}$
- $\{\mathtt{a}^p\mathtt{b}^q \;\mid\; p$ is odd $\}$.
- An automaton over $\{\mathtt{a}, \mathtt{b}, \mathtt{c}\}$ recognizing the string that miss at least one letter.

Some applications

- $L = \{ \mathtt{a}\, w \mathtt{z} \mid w \in \Sigma^* \}$
- $\{\mathtt{a}^p \mathtt{b}^q \mid p \text{ is odd } \}$.
- An automaton over $\{\mathtt{a}, \mathtt{b}, \mathtt{c}\}$ recognizing
  the string that miss at least one letter. (The union of $\{\mathtt{a}, \mathtt{b}\}^*$, $\{\mathtt{b}, \mathtt{c}\}*$
  and $\{\mathtt{c}, \mathtt{a}\}^*$).

BASIC AND REGULAR LANGUAGES

Basic languages

- Fix Σ. The ***basic Σ-languages*** are generated by:

Basic languages

- Fix Σ. The ***basic Σ-languages*** are generated by:
  - All finite languages

Basic languages

- Fix Σ. The ***basic Σ-languages*** are generated by:
  - All finite languages
  - Obtained by set operations:
    If L, L' are basic then so are
    $L \cup L'$, $L \cap L'$, and $\bar{L} = \Sigma^* - L$

Basic languages

- Fix Σ. The ***basic Σ-languages*** are generated by:
  - All finite languages
  - Obtained by set operations:
    If L, L' are basic then so are
    $L \cup L'$, $L \cap L'$, and $\bar{L} = \Sigma^* - L$
  - Obtained by language operations:
    If L, L' are basic then so are $L \cdot L'$ and L^* .

Regular languages

- The collection of ***regular languages*** is generated like the basic languages, but with more frugality.

- We shall see that every basic language is regular, but the frugality of regular languages allows an economy of efforts and notations.

- The generative rules for regular languages:

  - Basis: $\emptyset$, $\{\varepsilon\}$, and $\{\sigma\}$ for each $\sigma \in \Sigma^*$.
  - Set operation: If L and L' are regular then so is $L \cup L'$.
  - Language operations: If L and L' are regular, then so are $L \cdot L'$ and L^*.

Every regular language is basic

- Proof by induction on the definition fo regular language.
- The initial regular languages are all finite,
  so they are all initial basic languages.
- If regular languages L, L' are basic,
  then their union, concatenation and star are also basic,
  since the union and concatenation of basic languages are basic.

Regular expressions

- Aren't we all bored and tired of writing all these braces?
- We can keep track of the generative process by simple road-maps, called ***regular expressions***.
- Given Σ, the ***regular expressions over*** Σ are generated by:
  - The languages $\emptyset$, $\{\varepsilon\}$ and $\{\sigma\}$ are named by $\boldsymbol{\emptyset}, \boldsymbol{\varepsilon}$, and $\boldsymbol{\sigma}$.
  - If L, L' are named by α, α' then $L \cup L'$ is named by $(\alpha)\cup(\alpha')$, $L \cdot L'$ by $(\alpha)\bullet(\alpha')$, and L^* by $(\alpha)^\star$

Decoding reg exp

- Formally, the function $\mathcal{L}$
  from regular expressions to regular languages
  is defined by recurrence on the definition of reg exps.

- Base.

$$
\begin{aligned}
\mathcal{L}(\boldsymbol{\emptyset}) &= \emptyset \\
\mathcal{L}(\boldsymbol{\varepsilon}) &= \{\varepsilon\} \\
\mathcal{L}(\boldsymbol{\sigma}) &= \{\sigma\} \quad (\sigma \in \Sigma)
\end{aligned}
$$

- Recurrence cases:

$$
\begin{aligned}
\mathcal{L}(\alpha \cup \beta &= \mathcal{L}(\alpha) \cup \mathcal{L}(\beta) \\
\mathcal{L}(\alpha \bullet \beta &= \mathcal{L}(\alpha) \cdot \mathcal{L}(\beta) \\
\mathcal{L}(\alpha^{\star}) &= \mathcal{L}(\alpha)^{*}
\end{aligned}
$$

THE GRAND REGULAR UNITY

What makes automata and regularity so central

- We have three imporance language properties.
  - Basic
  - Recognized
  - Regular
- Each is consequential,
  and their equivalence demonstrates unity and coherence

Uniting three definitions

- We'll see that the following properties of languages are equivalent.
  - L is basic
  - L is recognized by an automaton
  - L is regular
  - L has finitely many residues
- The proofs are much easier using a broader notion of an automaton, called ***nondeterministic automaton*** (NFAs).
- To avoid ambiguity, we'll refer to automata as ***deterministic automata*** (DFAs).
- Of course, we'll need to show that a language is recognized by an NFA iff it is recognized by a DFA.

NONDETERMINISTIC AUTOMATA

The concatenation of recognized languages

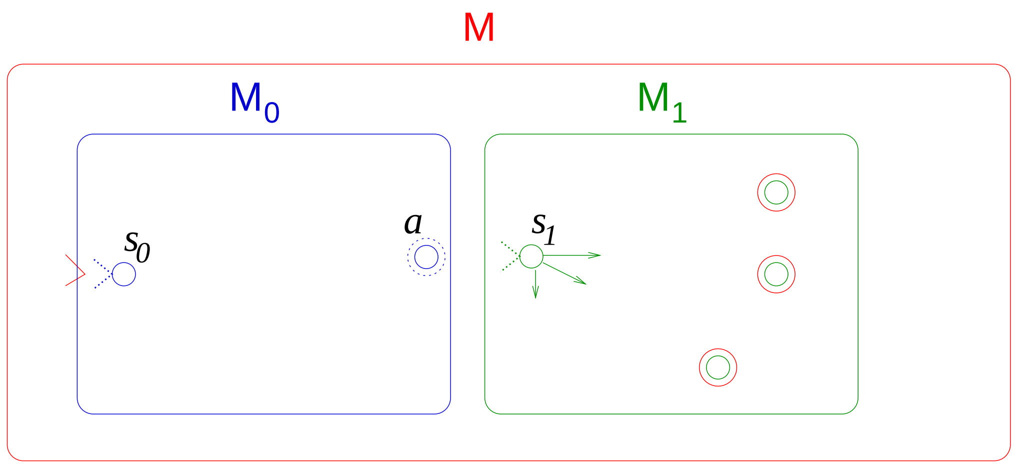
- We proved: If L, L' are recognized then so are $L \cup L'$, $L \cap L'$ and $L - L'$.

The concatenation of recognized languages

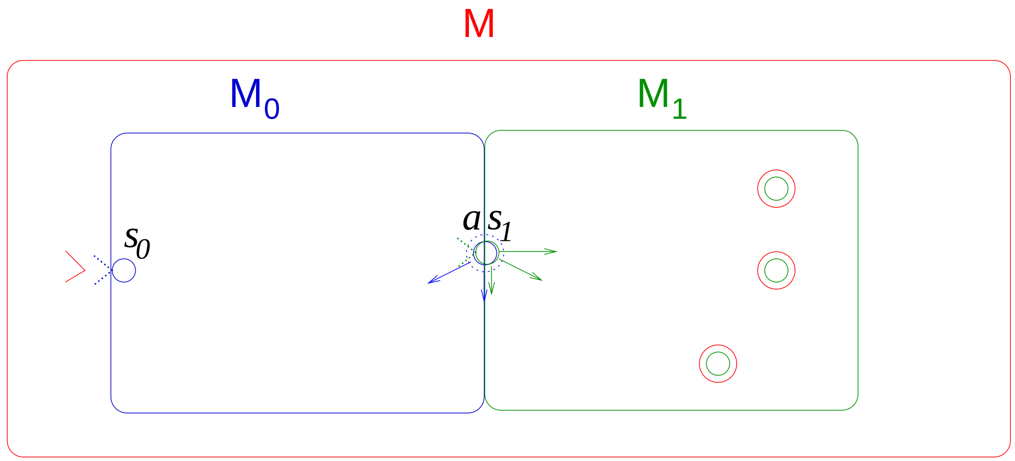
- We proved: If L, L' are recognized then so are $L \cup L'$, $L \cap L'$ and $L - L'$.
- Concatenation?
  Given automata M and M' recognizing L and L'
  construct automaton K recognizing $L \cdot L'$.

The concatenation of recognized languages

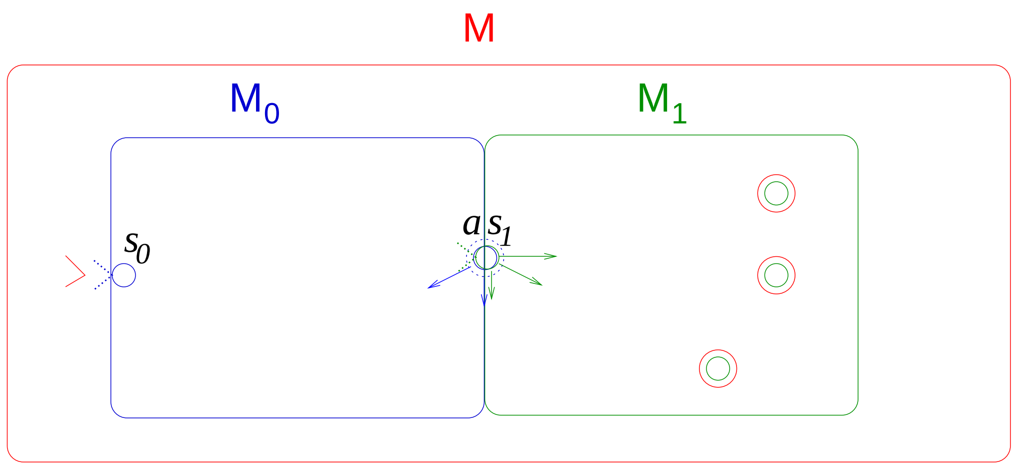
- We proved: If L, L' are recognized then so are $L \cup L'$, $L \cap L'$ and $L - L'$.
- Concatenation?
  Given automata M and M' recognizing L and L'
  construct automaton K recognizing $L \cdot L'$.



Trying to make this work

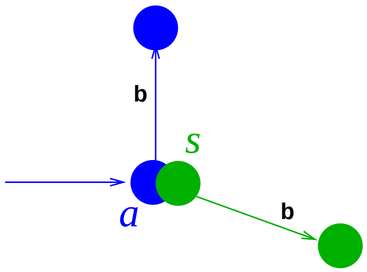


Trying to make this work



- Problem: Can't coalesce a and σ_1 :

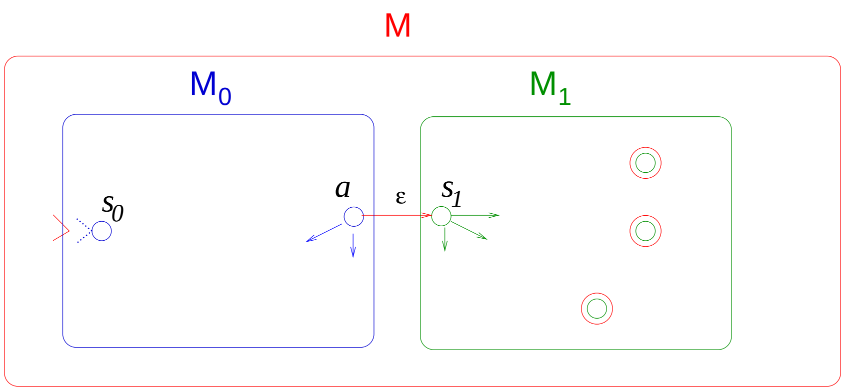
  They might have conflicting transitions rules:



And computation might proceed back and forth between M_0 and M_1 .

Spontaneous transitions

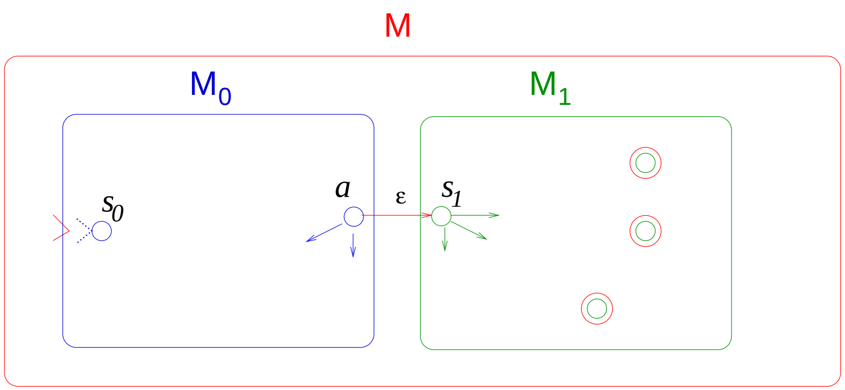
- We can force the computation to proceed from M_0 to M_1 by allowing spontaneous transitions between states, $q \longrightarrow p$ without any symbol read.



- We call these **epsilon-transitions**, in analogy to the notation $q \xrightarrow{w} p$.

Spontaneous transitions

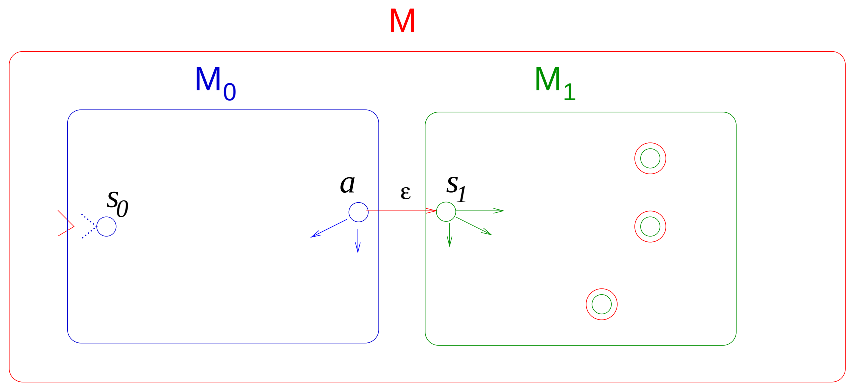
- We can force the computation to proceed from M_0 to M_1 by allowing spontaneous transitions between states, $q \longrightarrow p$ without any symbol read.



- We call these **epsilon-transitions**, in analogy to the notation $q \xrightarrow{w} p$.

Spontaneous transitions

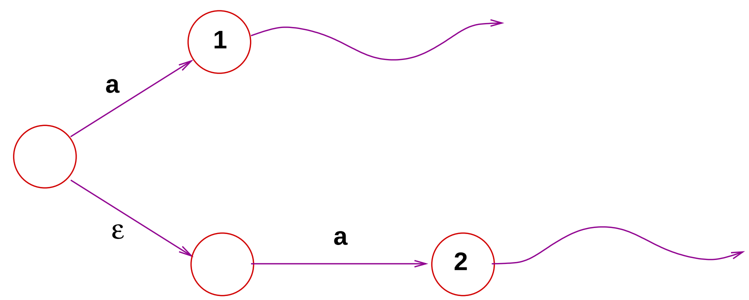
- We can force the computation to proceed from M_0 to M_1 by allowing spontaneous transitions between states, $q \rightarrow p$ without any symbol read.



- We call these **epsilon-transitions**, in analogy to the notation $q \xrightarrow{w} p$.

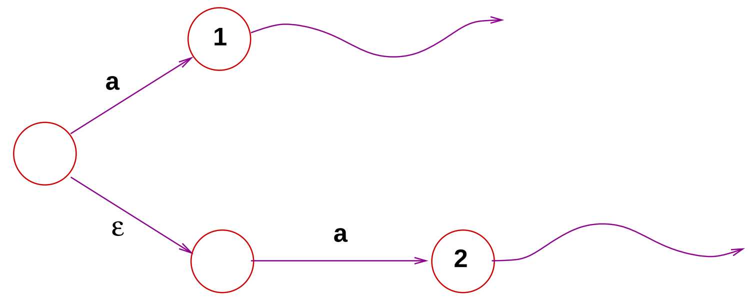
Nondeterminism

- *ε*-transitions yield “ambiguous” computation:



Nondeterminism

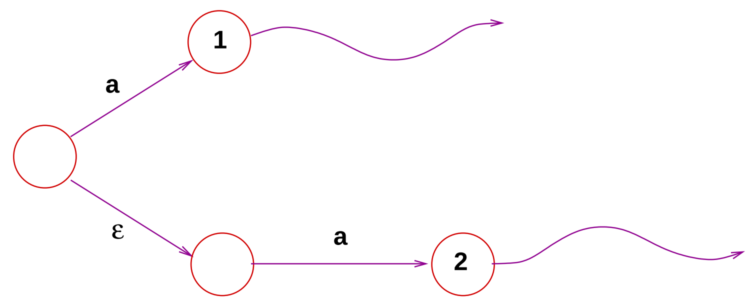
- ε-transitions yield "ambiguous" computation:



- So we might as well allow non-univalent (AKA ***nondeterministic***) transition rules.

Nondeterminism

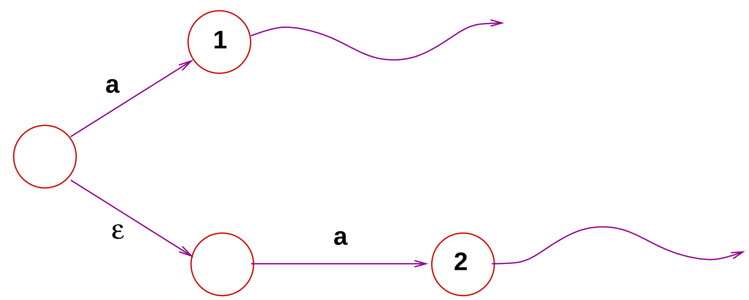
- ε-transitions yield "ambiguous" computation:



- So we might as well allow non-univalent (AKA ***nondeterministic***) transition rules.
- This does not correspond to normal hardware behavior, but:
  - The notion is important elsewhere

Nondeterminism

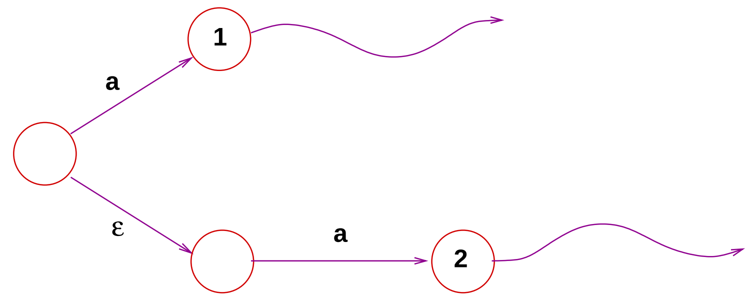
- ε-transitions yield "ambiguous" computation:



- So we might as well allow non-univalent (AKA ***nondeterministic***) transition rules.
- This does not correspond to normal hardware behavior, but:
  - The notion is important elsewhere
  - It can be simulated by ε-transitions, which do model natural phenomena; and

Nondeterminism

- ε-transitions yield “ambiguous” computation:

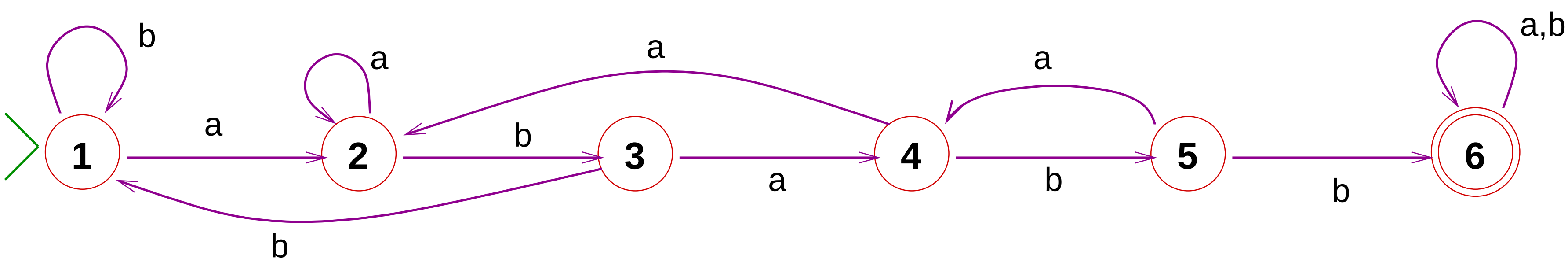


- So we might as well allow non-univalent (AKA ***nondeterministic***) transition rules.
- This does not correspond to normal hardware behavior, but:
  - The notion is important elsewhere
  - It can be simulated by ε-transitions, which do model natural phenomena; and
  - It is algorithmically natural, as we see next.

AUTOMATA AS ON-LINE ALGORITHMS

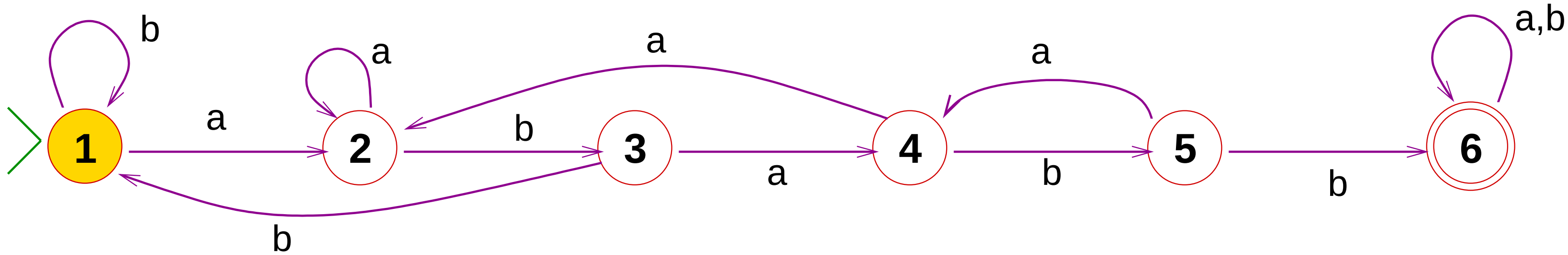
Automata as on-line algorithms

- Automata can be viewed as efficient real time algorithms, which move pointers (or “tokens”) around.
- An automaton to recognize the presence of `ababb`:



The operation visualized

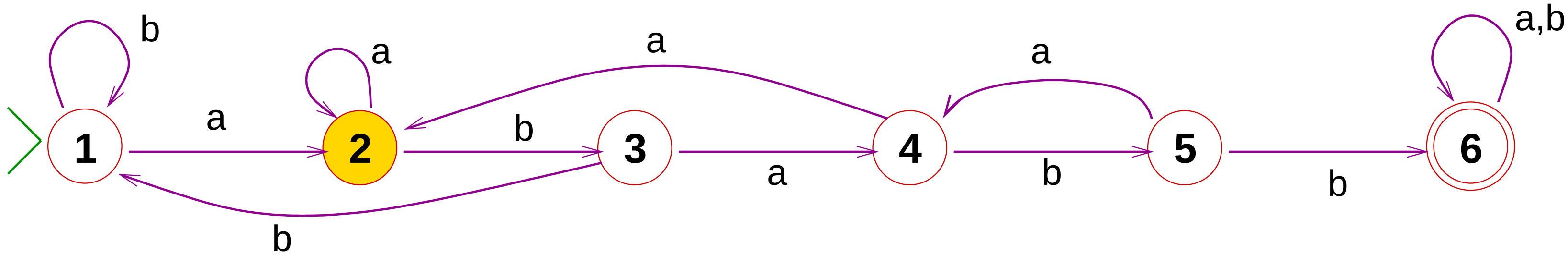
- The automaton's operation can be visualized by moving a token designating the current state.



a̲ b a b a b b a

The operation visualized

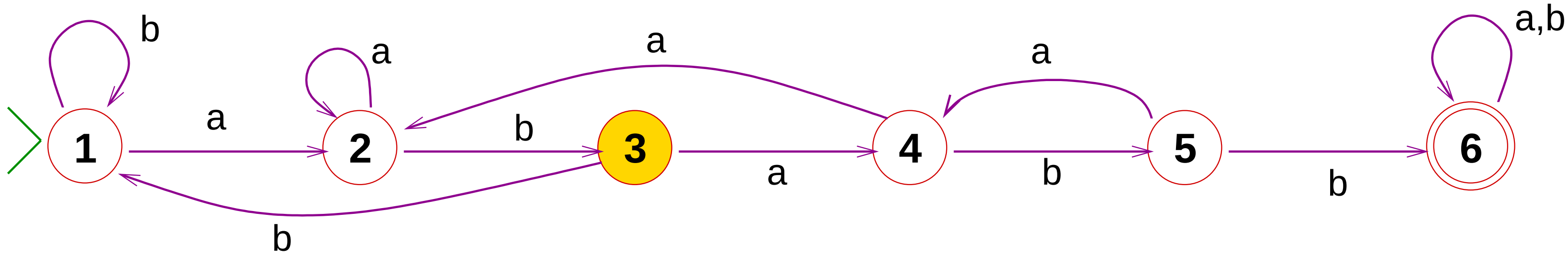
- The automaton's operation can be visualized by moving a token designating the current state.



a <u>b</u> a b a b b a

The operation visualized

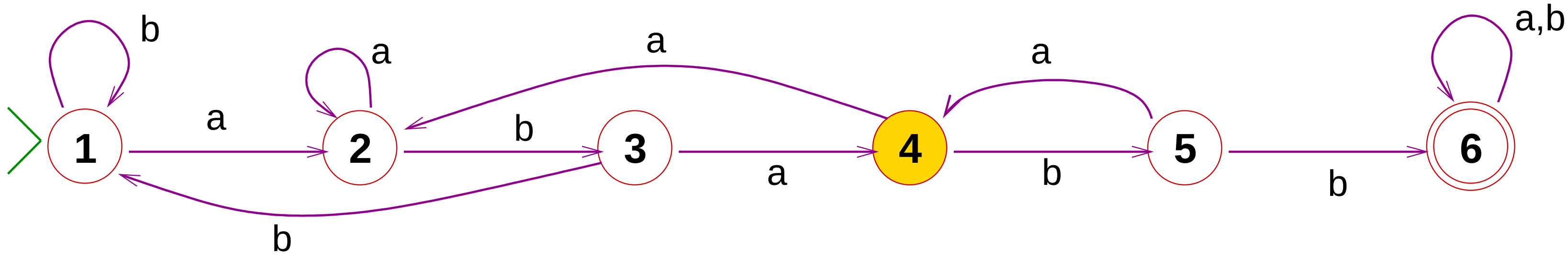
- The automaton's operation can be visualized by moving a token designating the current state.



a b <u>a</u> b a b b a

The operation visualized

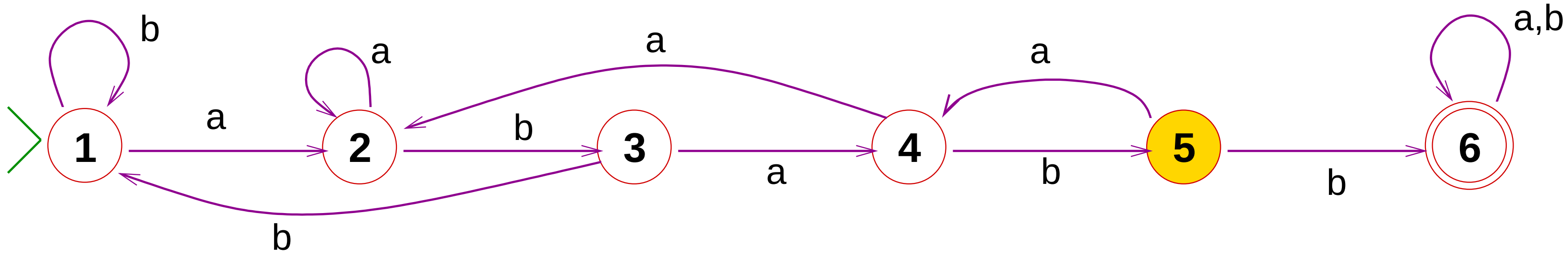
- The automaton's operation can be visualized by moving a token designating the current state.



a b a <u>b</u> a b b a

The operation visualized

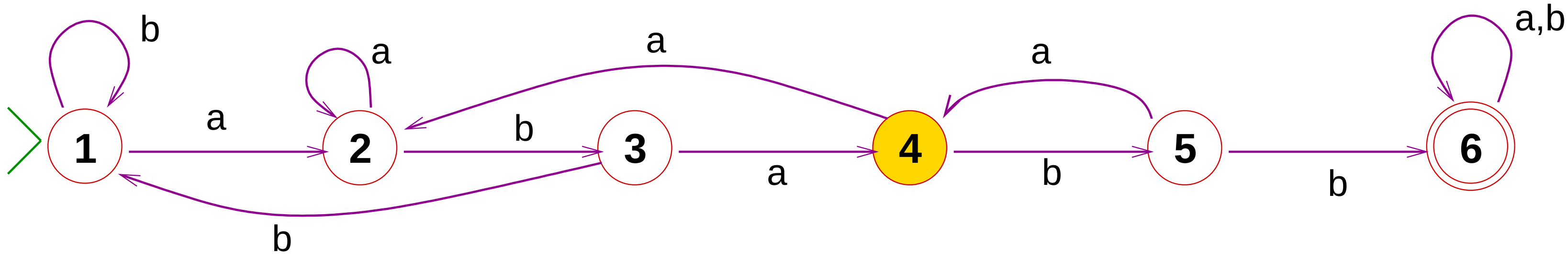
- The automaton's operation can be visualized by moving a token designating the current state.



a b a b a b b a

The operation visualized

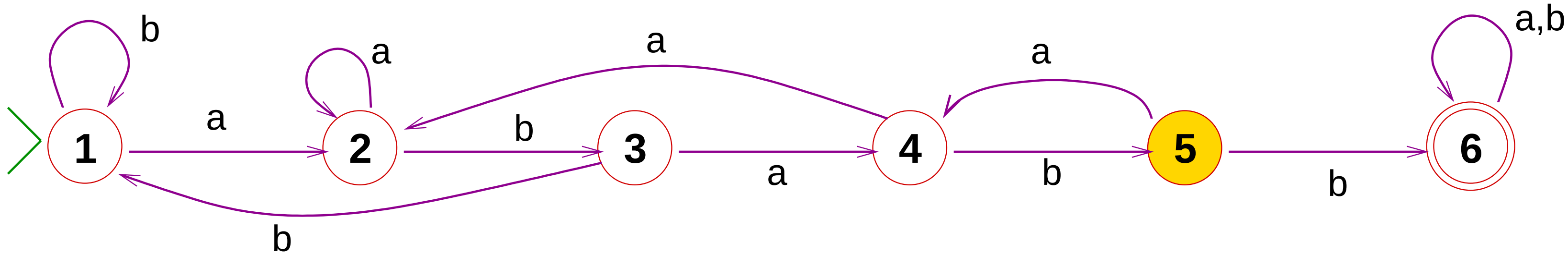
- The automaton's operation can be visualized by moving a token designating the current state.



a b a b a <u>b</u> b a

The operation visualized

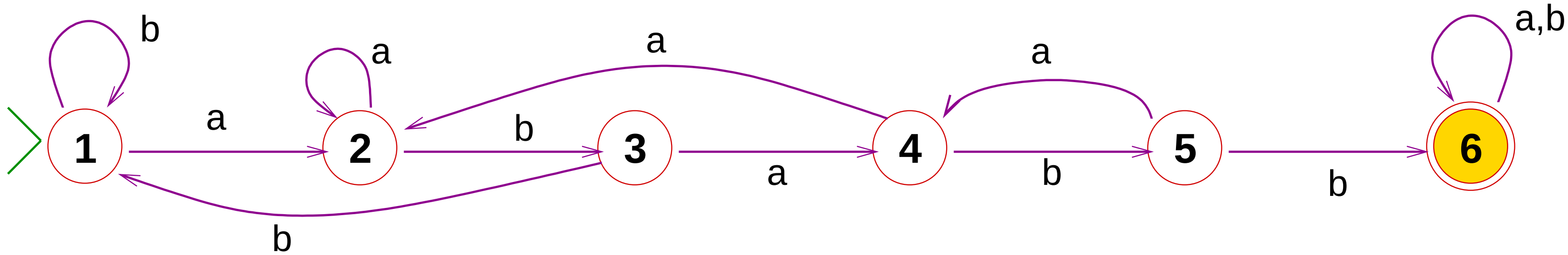
- The automaton's operation can be visualized by moving a token designating the current state.



a b a b a b <u>b</u> a

The operation visualized

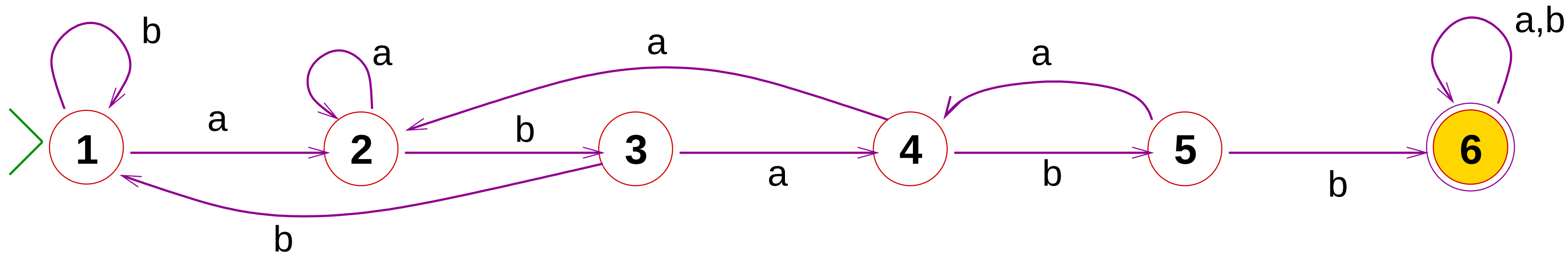
- The automaton's operation can be visualized by moving a token designating the current state.



a b a b a b b <u>a</u>

The operation visualized

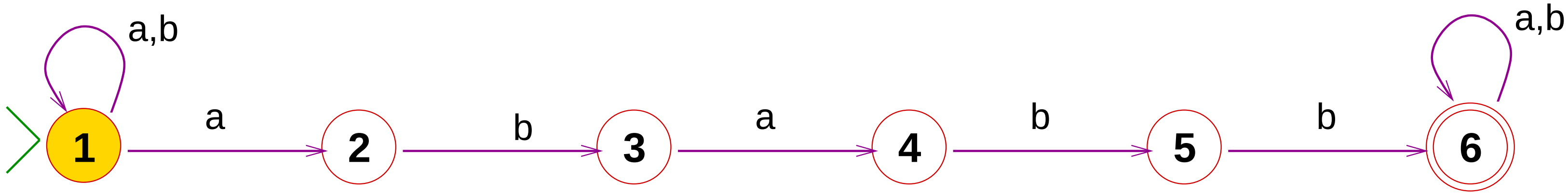
- The automaton's operation can be visualized by moving a token designating the current state.



a b a b a b b a _

An alternative, with token rules relaxed

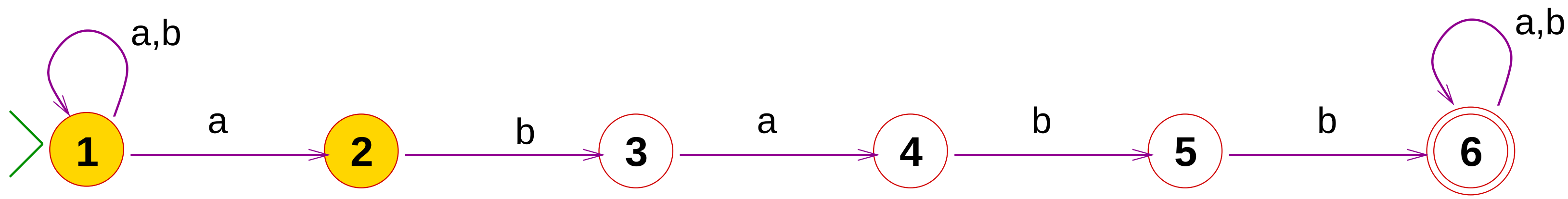
- Here we have ambiguities at the start and end of the chain.



a b a b a b b a

An alternative, with token rules relaxed

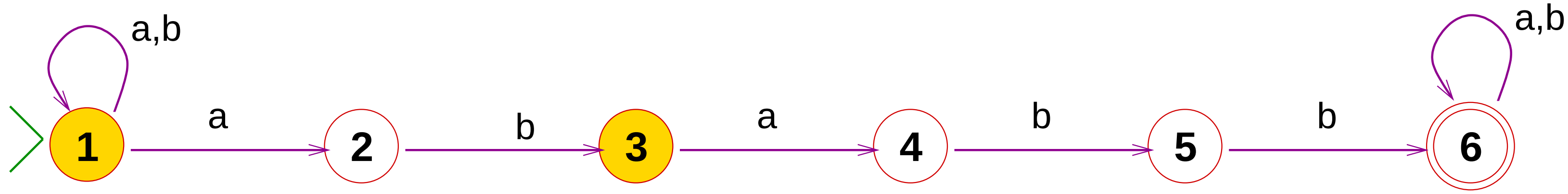
- There are options for the "current state".



a <u>b</u> a b a b b a

An alternative, with token rules relaxed

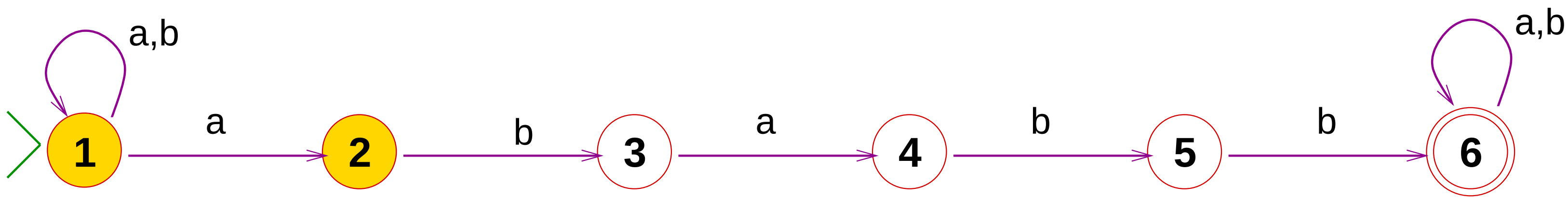
- There are options for the "current state".



a b <u>a</u> b a b b a

An alternative, with token rules relaxed

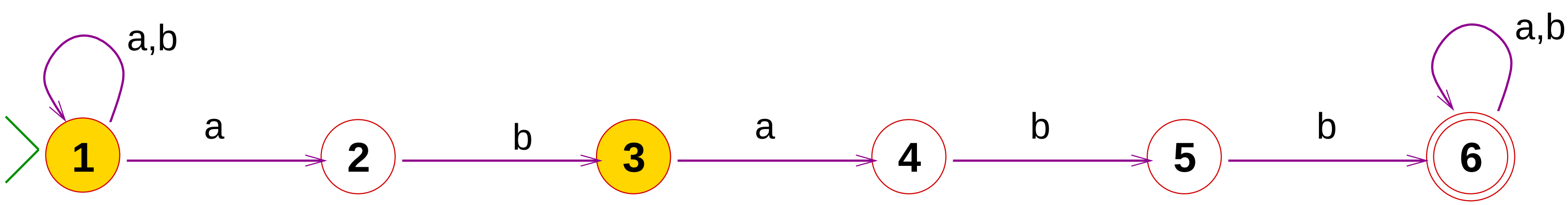
- There are options for the “current state”.



a b a <u>b</u> a b b a

An alternative, with token rules relaxed

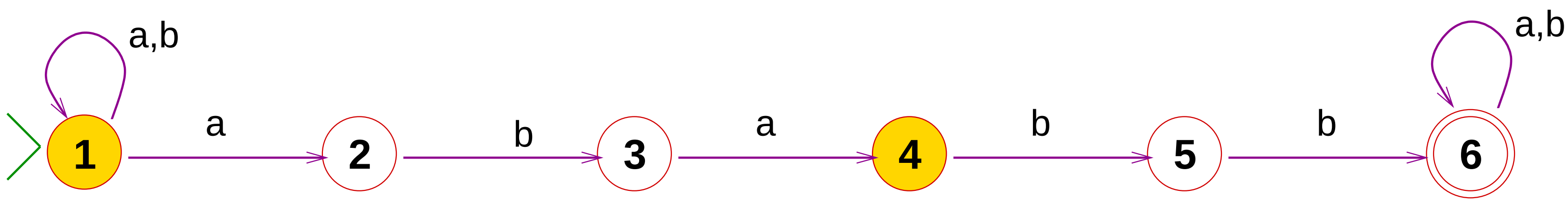
- There are options for the "current state".



a b a b a b b a

An alternative, with token rules relaxed

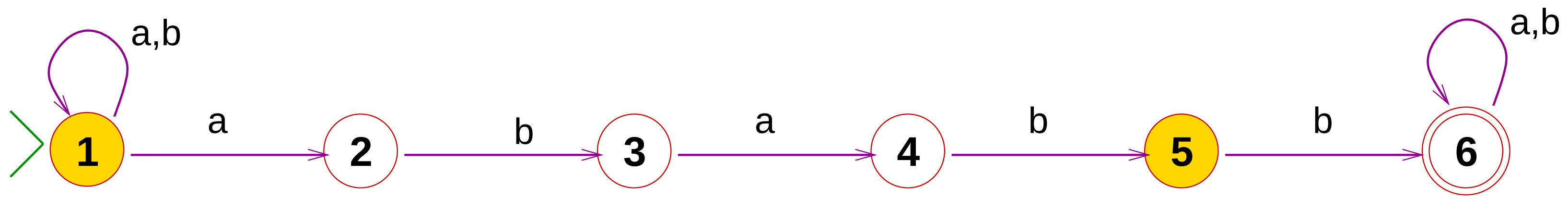
- There are options for the “current state”.



a b a b a <u>b</u> b a

An alternative, with token rules relaxed

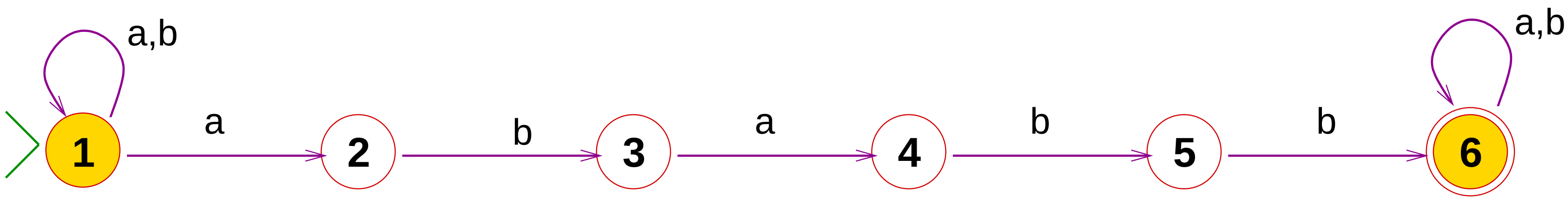
- There are options for the “current state”.



a b a b a b <u>b</u> a

An alternative, with token rules relaxed

- There are options for the “current state”.



a b a b a b b a

Non-deterministic automata

A non-deterministic automaton over Σ:

- Finite (non-empty) set Q of states
- Start state s and accepting states $A \subseteq Q$
- Transition mapping: $\delta : (Q \times \Sigma_\epsilon) \Rightarrow Q$
- Here $\Sigma_\epsilon = \Sigma \cup \{\varepsilon\}$
- Still using the notation $q \xrightarrow{\sigma} p$ for $\langle q, \sigma, p \rangle \in \delta$
- But $q \xrightarrow{\epsilon} p$ is also an option.

Computation state-traces

- If $w = \sigma_1 \cdot \sigma_2 \cdots \cdot \sigma_n$ where $\sigma_i \in \Sigma_\varepsilon$,
  and $q \xrightarrow{\sigma_1} r_1 \xrightarrow{\sigma_2} r_2 \quad \cdots \quad r_{n-1} \xrightarrow{\sigma_n} p$
  then $q \overset{w}{\Longrightarrow} p$.

Computation state-traces

- If $w = \sigma_1 \cdot \sigma_2 \cdot \cdots \cdot \sigma_n$ where $\sigma_i \in \Sigma_\varepsilon$,
  and $q \xrightarrow{\sigma_1} r_1 \xrightarrow{\sigma_2} r_2 \quad \cdots \quad r_{n-1} \xrightarrow{\sigma_n} p$
  then $q \overset{w}{\Longrightarrow} p$.
- The sequence of states

$$q \;\; r_1 \;\; r_2 \;\; \cdots \;\; r_{n-1} \;\; p$$

  as above is a ***state-trace*** of the NFA for input w.

Generative definition of $q \overset{w}{\Longrightarrow} p$

- **Base.** $q \overset{\epsilon}{\to} q$ for all $q \in Q$.
- **Step.** If $q \overset{\sigma}{\to} p$ by the NFA's transition,
  and $p \overset{w}{\Longrightarrow} r$ has been generated already (where $\sigma \in \Sigma_\epsilon$) then $q \overset{\sigma \cdot w}{\Longrightarrow} r$.

Acceptance by an NFA

- M ***accepts*** a string $w \in \Sigma^*$ if $s \stackrel{w}{\Longrightarrow} A$.

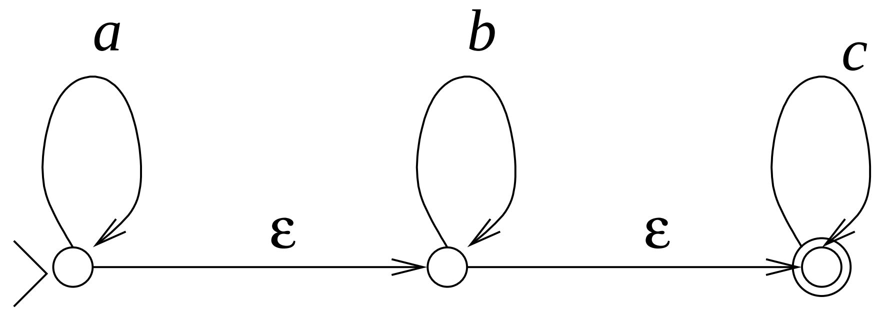
Acceptance by an NFA

- M ***accepts*** a string $w \in \Sigma^*$ if $s \stackrel{w}{\Longrightarrow} A$.
- This dfn is like for DFAs, but now
  1. A string w is accepted if there is ***some*** state-trace for $s \stackrel{w}{\Longrightarrow} A$.
  2. A "lucky trace" may include ε-transitions.

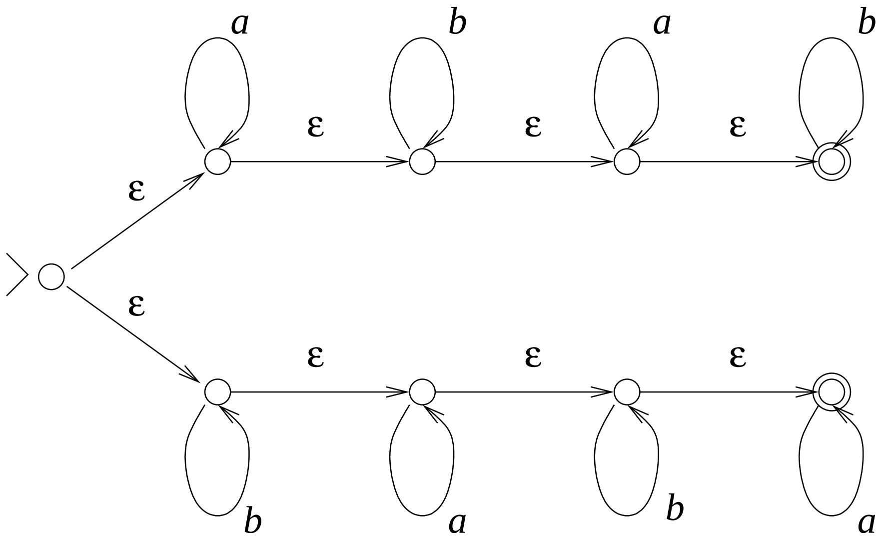
Acceptance by an NFA

- M ***accepts*** a string $w \in \Sigma^*$ if $s \overset{w}{\Longrightarrow} A$.
- This dfn is like for DFAs, but now
  1. A string w is accepted if there is ***some*** state-trace for $s \overset{w}{\Longrightarrow} A$.
  2. A "lucky trace" may include ε-transitions.
- The ***language recognized*** by M
  is the set of accepted strings.

Example: $\mathcal{L}(\texttt{a*b*c*})$

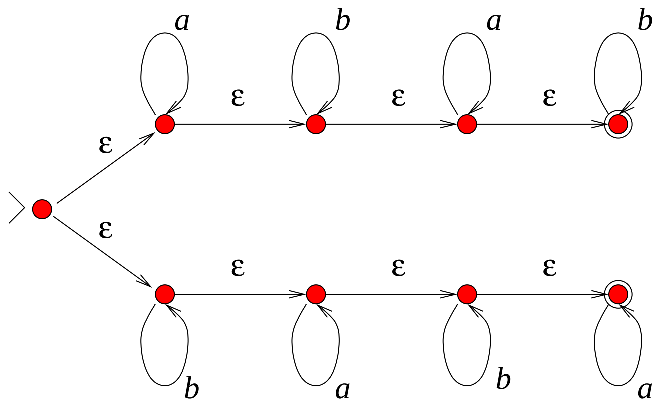


Recognizing $\mathcal{L}(a^*b^*a^*b^* \cup b^*a^*b^*a^*)$



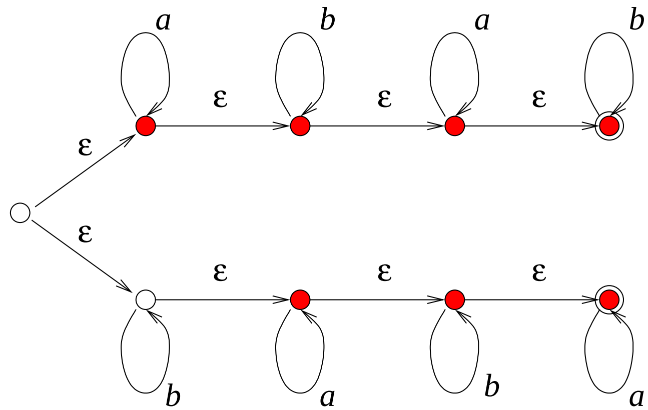
$a^* b^* a^* b^* \cup b^* a^* b^* a^*$

Recognizing $\mathcal{L}(a^*b^*a^*b^* \cup b^*a^*b^*a^*)$



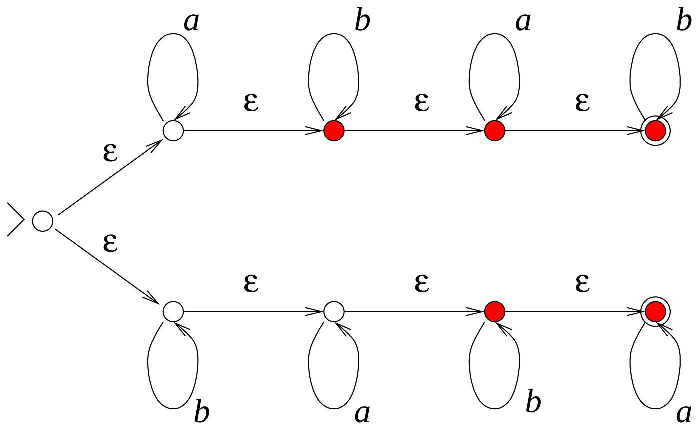
>abb

Recognizing $\mathcal{L}(a^*b^*a^*b^* \cup b^*a^*b^*a^*)$



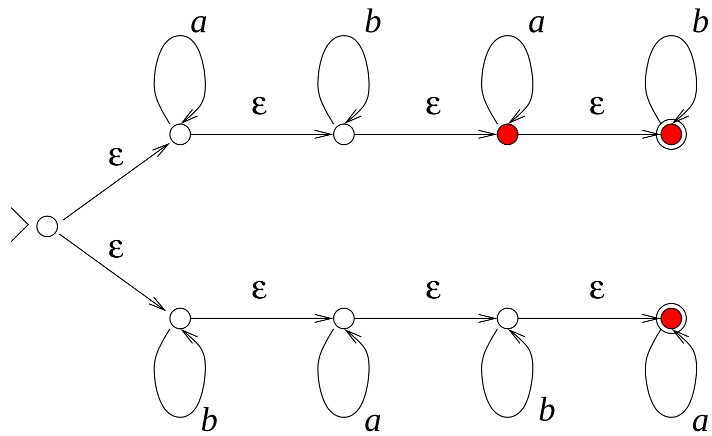
a>bb

Recognizing $\mathcal{L}(a^*b^*a^*b^* \cup b^*a^*b^*a^*)$



ab>b

Recognizing $\mathcal{L}(a^*b^*a^*b^* \cup b^*a^*b^*a^*)$



abb>

So the number of states is *reduced* with each step.

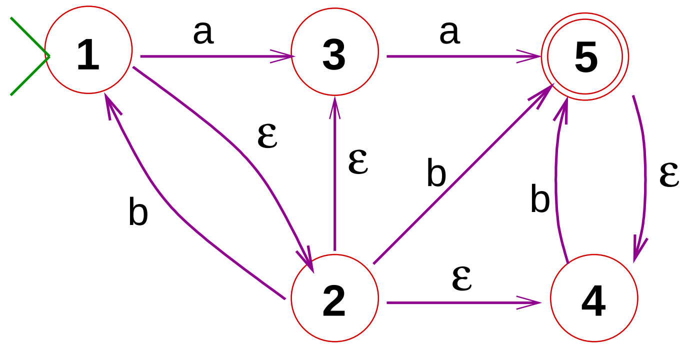
DFAs are special NFAs

- NFAs ***allow*** non-univalence, they don't require it!
- So *Every DFA is a special NFA,*
  where the transition mapping happens to be univalent

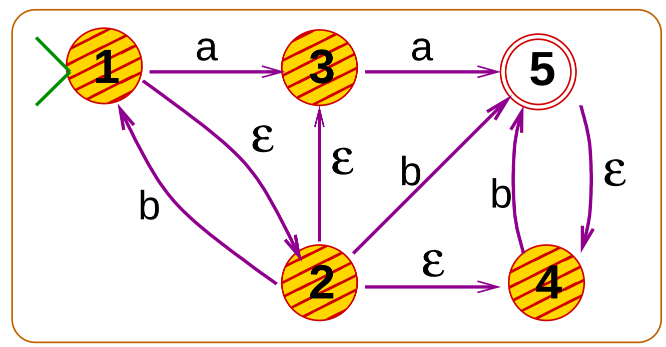
Converting NFAs to equivalent DFAs

An NFA-to-DFA coversion example

- Given an NFA N :

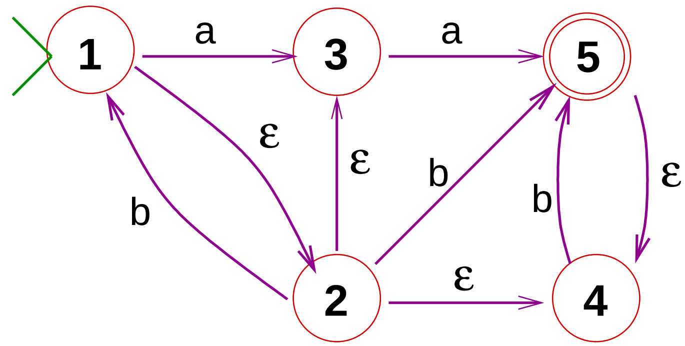


- Mark as "on" the states reachable on entry:

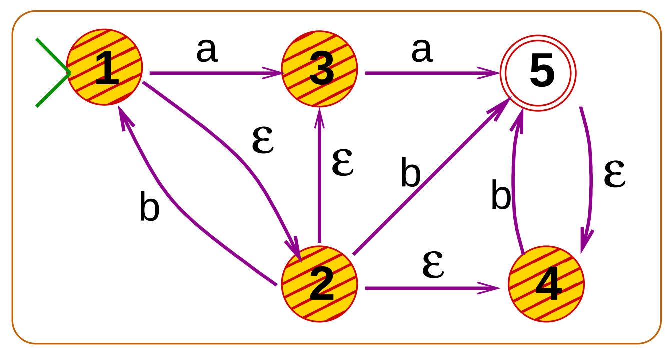


An NFA-to-DFA coversion example

- Given an NFA N :

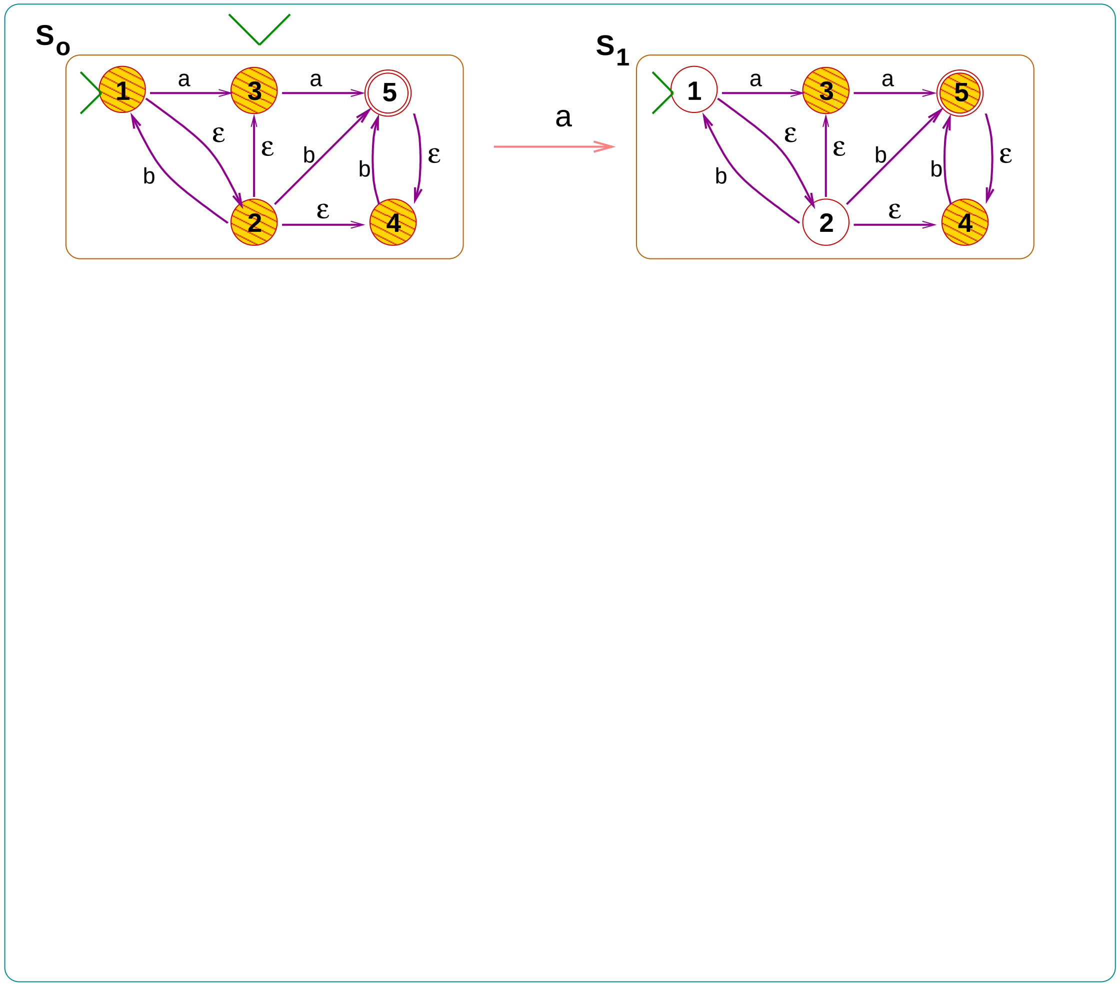


- Mark as “on” the states reachable on entry:

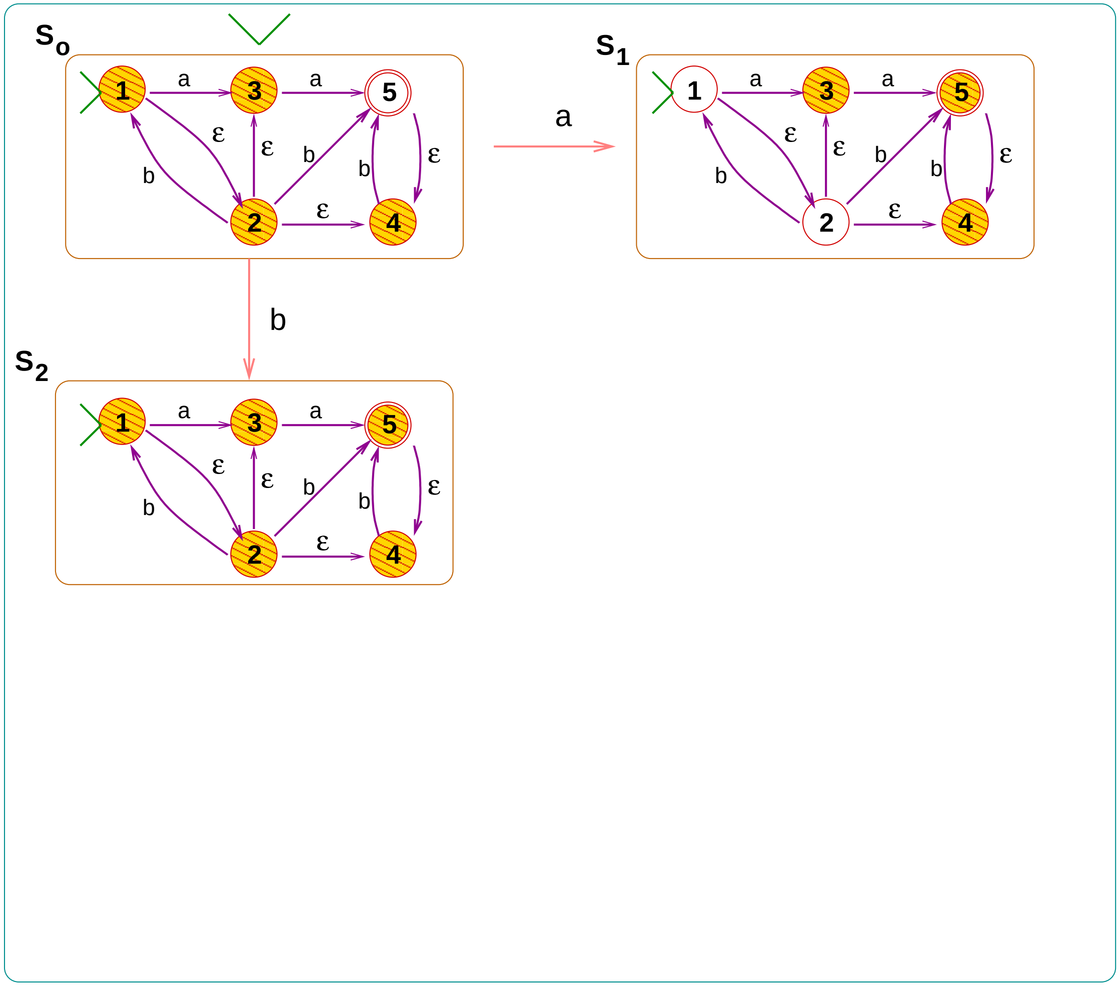


- This “super-state” is the ***start-state*** of our DFA.

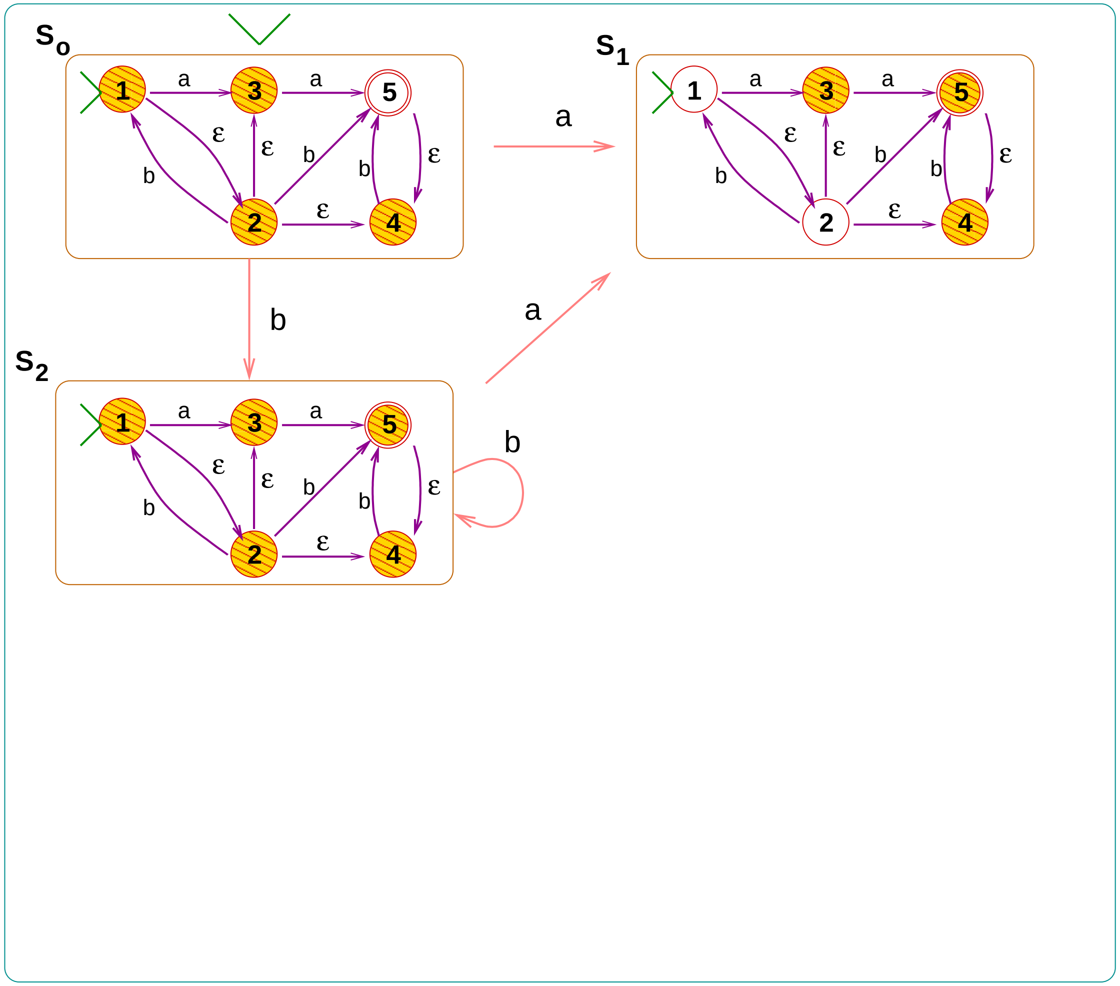
- The possible states on reading an `a` :



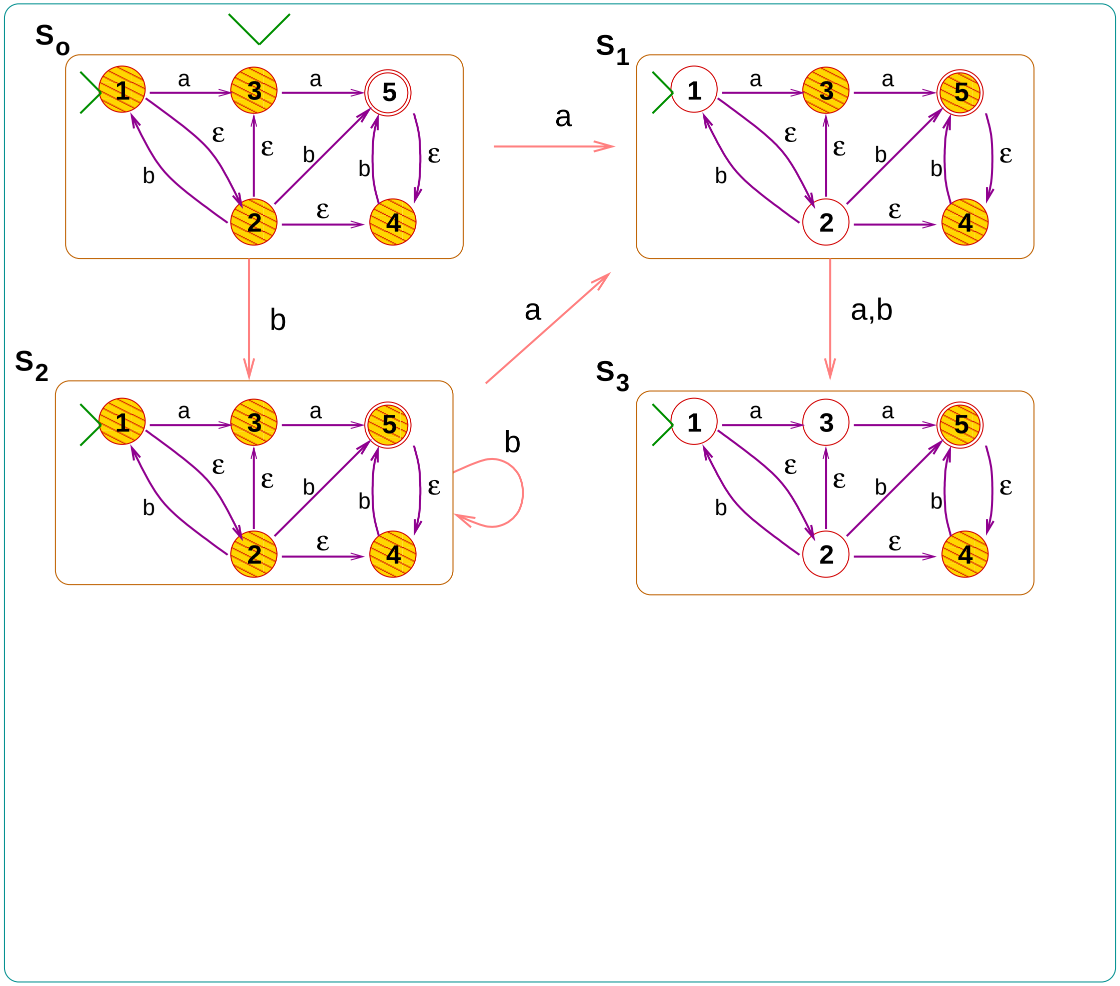
- Explore the super-states of reachable states:



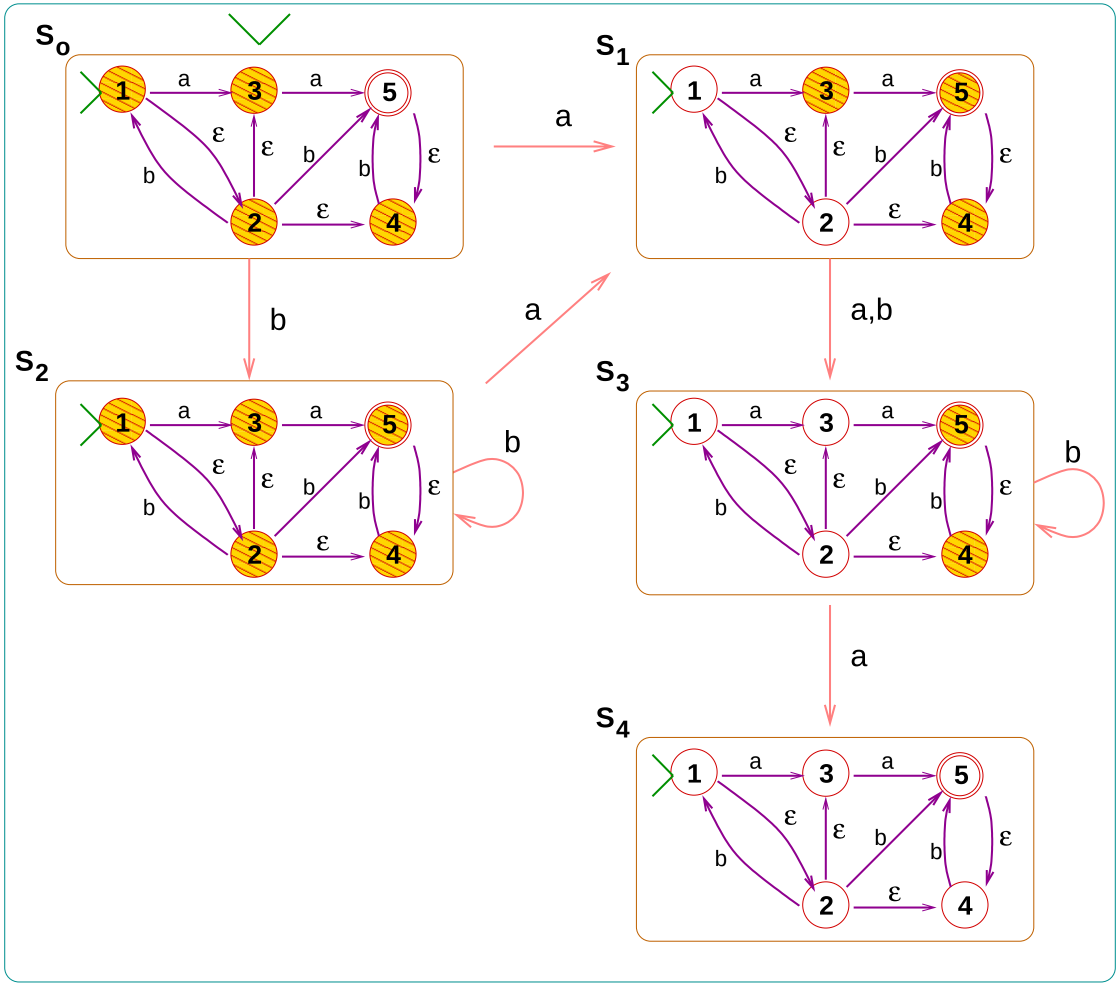
- Explore the super-states of reachable states:



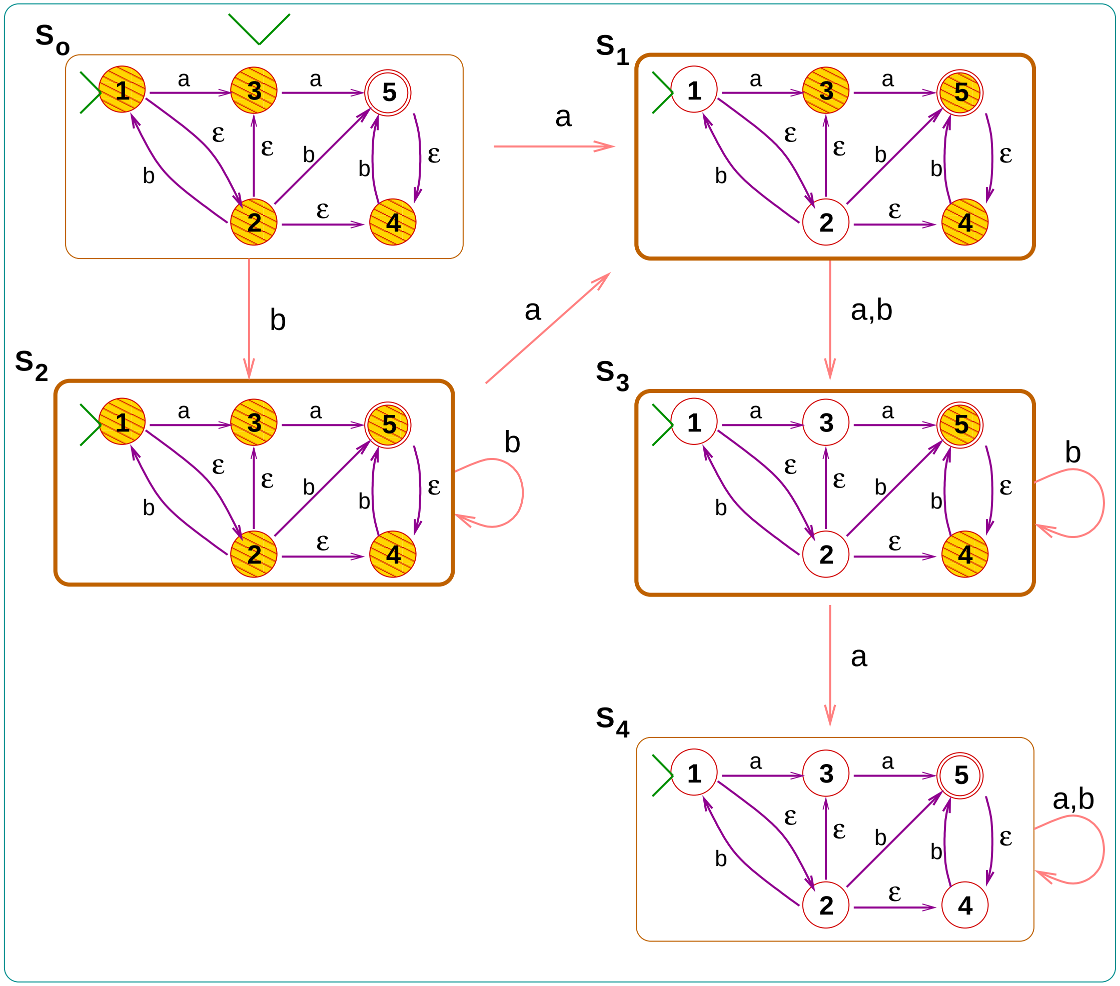
• Explore the super-states of reachable states:



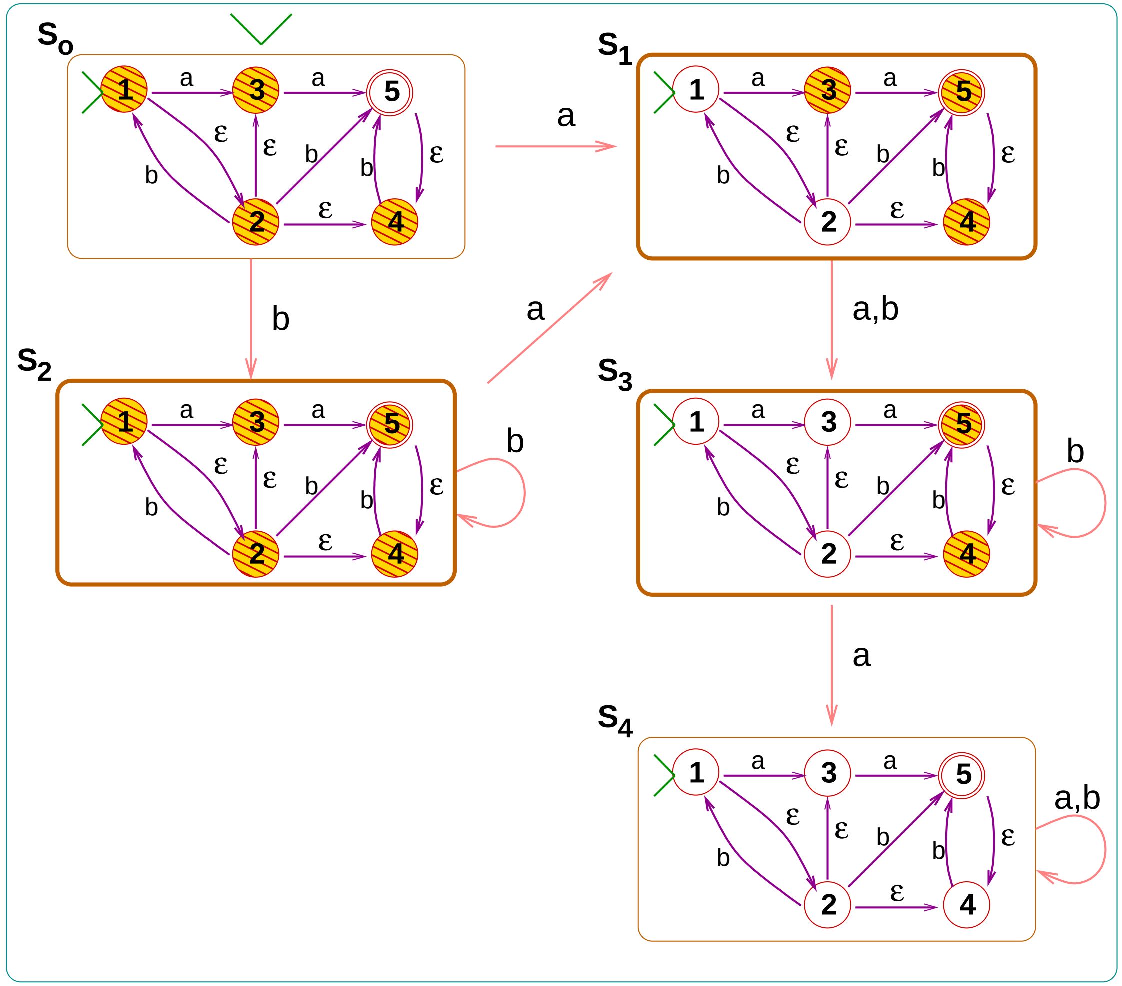
- Explore the super-states of reachable states:



- Explore the super-states of reachable states:



- A super-state is accepting if containing an accept-state:

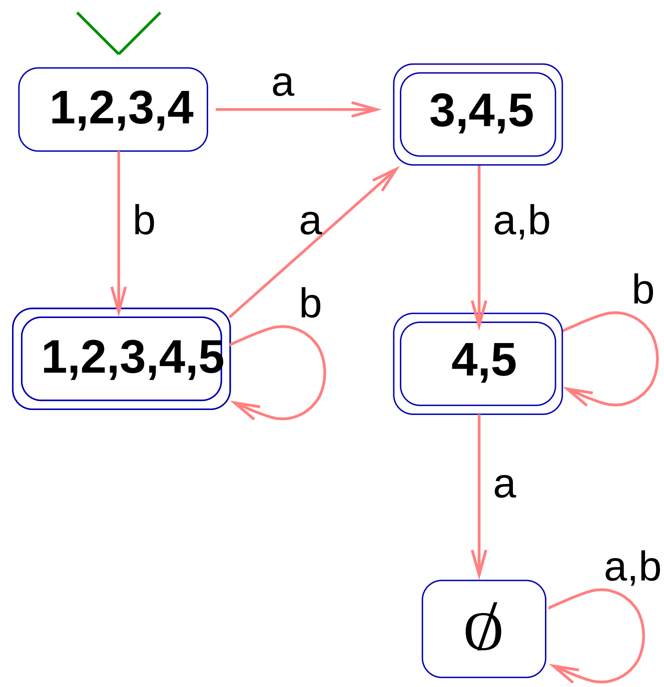


The resulting DFA

- We have constructed from the NFA N an equivalent DFA! Each state of the DFA obtained is a "super-state" of N's states:

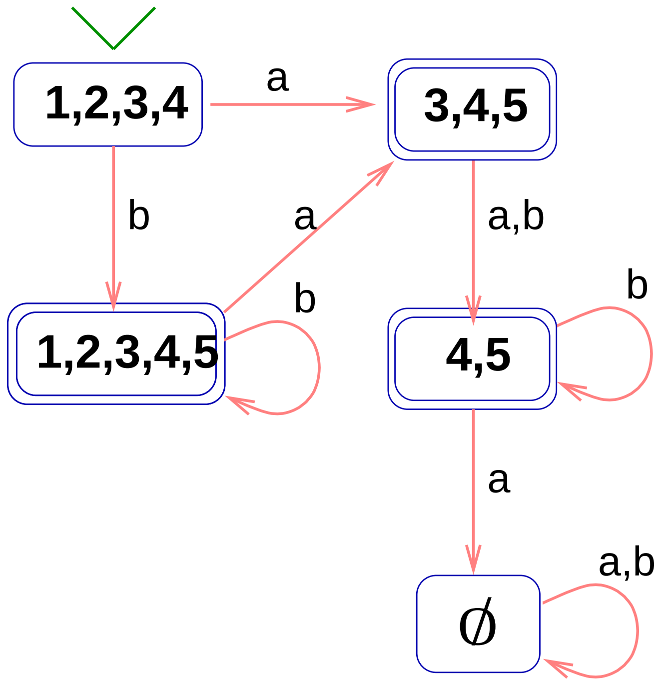
The resulting DFA

- We have constructed from the NFA N an equivalent DFA! Each state of the DFA obtained is a “super-state” of N's states:



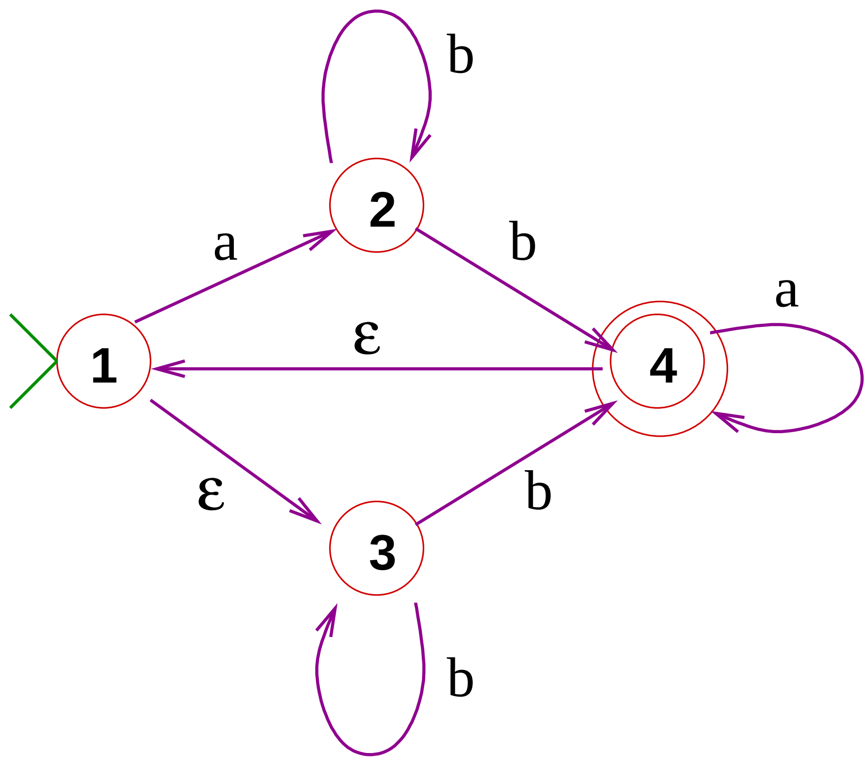
The resulting DFA

- We have constructed from the NFA N an equivalent DFA! Each state of the DFA obtained is a “super-state” of N's states:

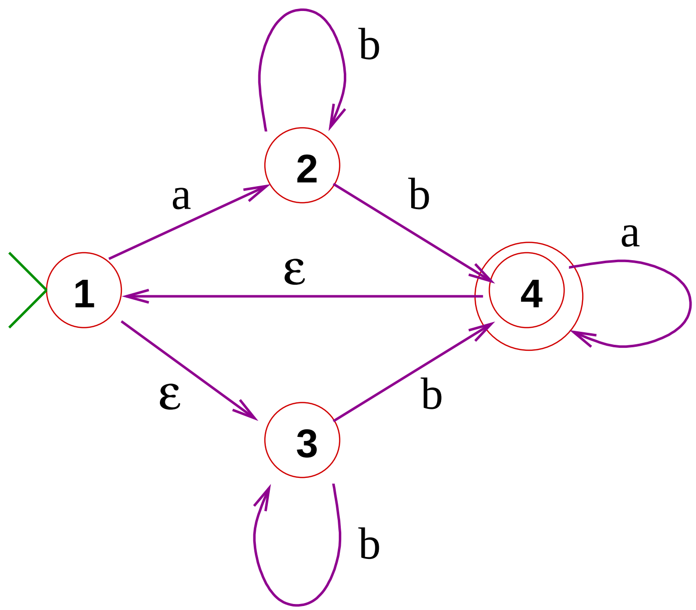


- We labeled here each state as the super-state it represents.

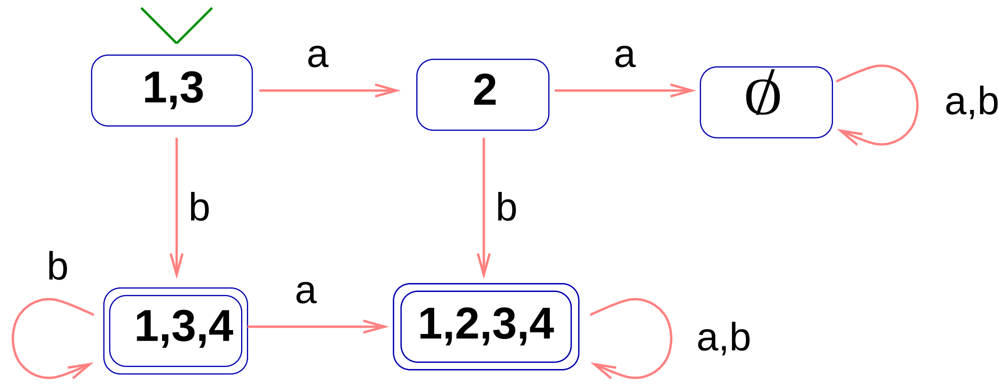
Another example



Another example

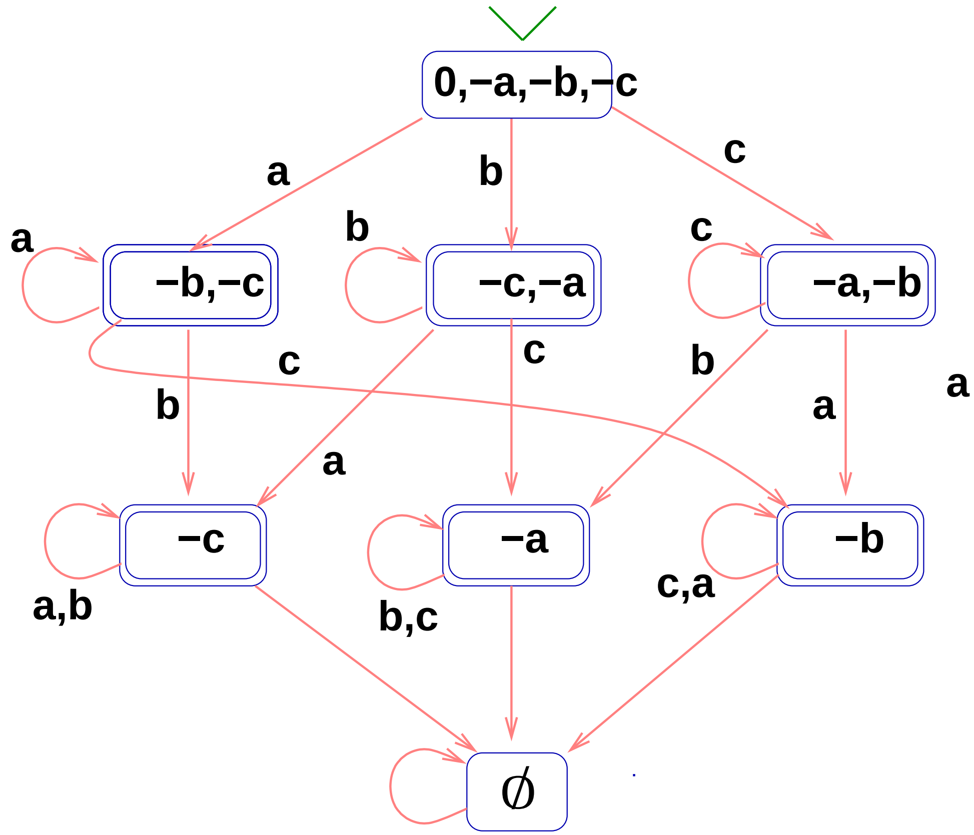


Another example

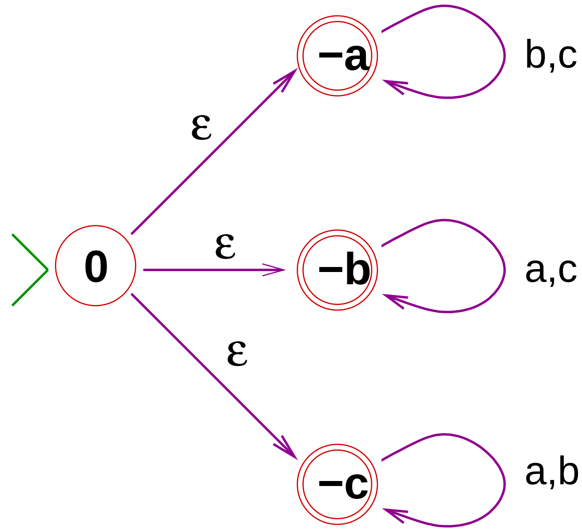


An exponential explosion

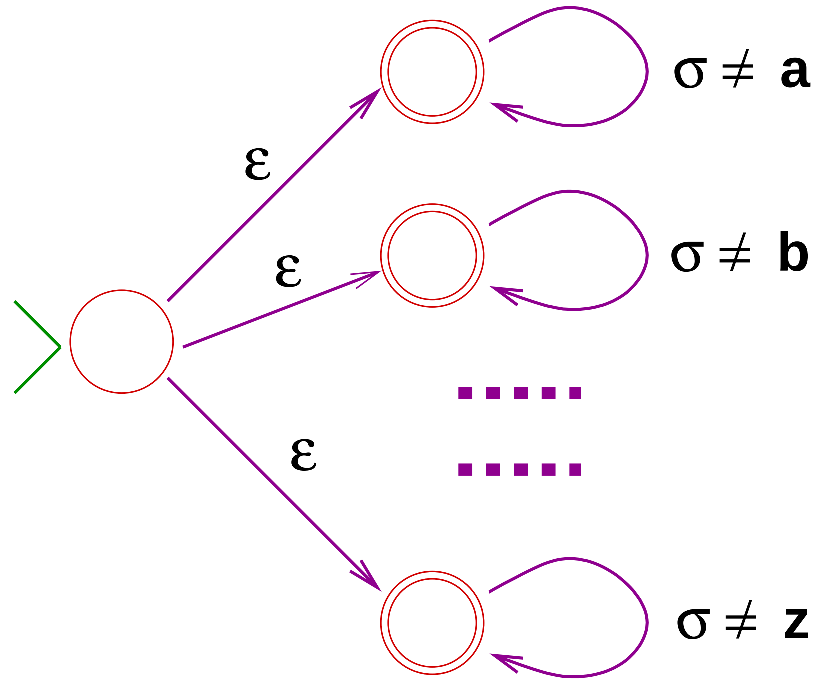
- If N has n states, then the DfA obtained
  may have up to 2^n states.
- Is that necessary?
- No! Consider the language of strings over $\{\mathtt{a}, \mathtt{b}, \mathtt{c}\}$ that miss at least one letter.
- The smallest DFA recognizing it is


0,−a,−b,−c
a
b
c
−b,−c
−c,−a
−a,−b
−c
−a
−b
a,b
b,c
c,a
∅

- But here is a 4-state NFA recognizing it:



- For “missed-som” language over the Latin alphabet
  the smalles recognizing automaton has $2^{26} > 67$ million states!
- But here is a 27 state NFA recognizing it:


ε
ε
ε
σ ≠ a
σ ≠ b
σ ≠ z

RECALL: Uniting three definitions

- We'll see that the following properties of languages are equivalent.
  - L is basic IMPLIES
  - L is recognized by an automaton
  - L is regular
  - L has finitely many residues

BASIC LANGUAGES ARE RECOGNIZED

Finite languages are recognized

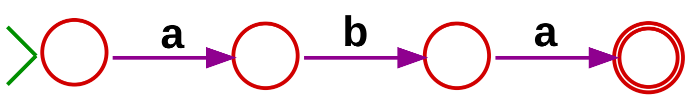
- $\emptyset$ is recognized by an NFA
  with one ***non-accepting*** state and no transitions.

Finite languages are recognized

- $\emptyset$ is recognized by an NFA
  with one ***non-accepting*** state and no transitions.
- $\{\varepsilon\}$ is recognized by an NFA
  with one ***accepting*** state and no transitions.

Finite languages are recognized

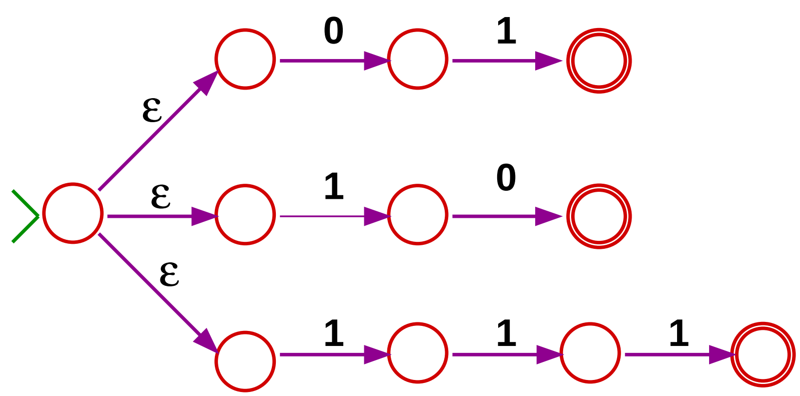
- $\emptyset$ is recognized by an NFA
  with one ***non-accepting*** state and no transitions.
- $\{\varepsilon\}$ is recognized by an NFA
  with one ***accepting*** state and no transitions.
- A string `aba` is recognized by the NFA



. Similarly for other strings.

- A finite language $\{w_1, \ldots, w_k\}$ is recognized by an NFA with ε-branching to k NFAs recognizing $\{w_1\}$ through $\{w_k\}$.

- A finite language $\{w_1, \ldots, w_k\}$ is recognized by an NFA with ε-branching to k NFAs recognizing $\{w_1\}$ through $\{w_k\}$.
- Example $\{01, 10, 111\}$ is recognized by



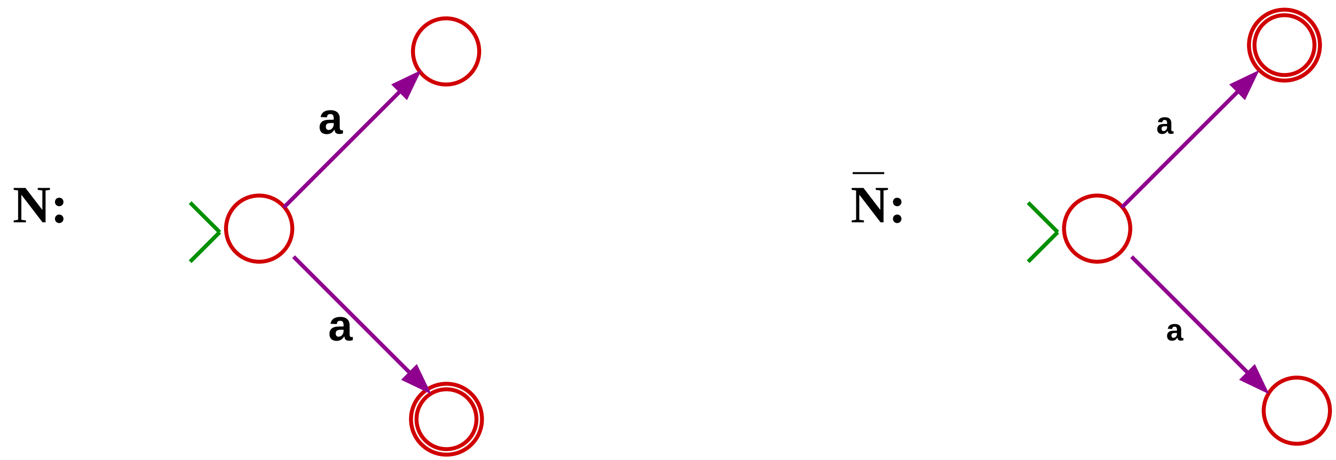
The complement of a recognized lang is recognized (reminder)

- As we have seen:
  If a language L is recognized by DFA M, then its complement is recognized by the DFA $\bar{M}$
  obtained by switching in M acceptance and non-acceptance.

The complement of a recognized lang is recognized (reminder)

- As we have seen:
  If a language L is recognized by DFA M, then its complement is recognized by the DFA $\bar{M}$
  obtained by switching in M acceptance and non-acceptance.

- Note: This idea doesn't work for NFAs:

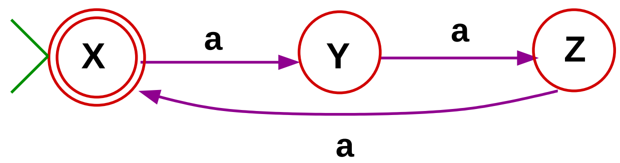


NFA N accepts a and so does $\bar{N}$.

The intersection of recognized languages is recognized (reminder)

Let $\Sigma = \{\mathtt{a}, \mathtt{b}\}$.

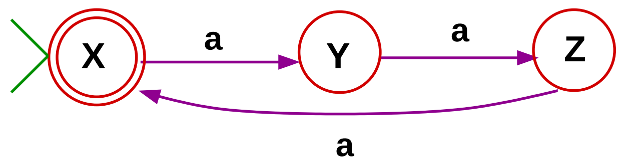
- Suppose M_3 recognizes $L_3 = \{w \in \Sigma^* \mid \#_a(w) = 0 \mod (3)\}$



The intersection of recognized languages is recognized (reminder)

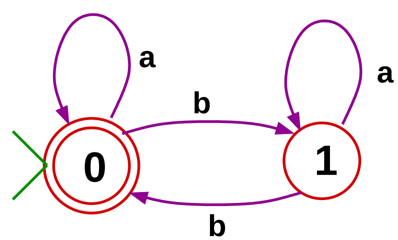
Let $\Sigma = \{\mathtt{a}, \mathtt{b}\}$.

- Suppose M_3 recognizes $L_3 = \{w \in \Sigma^* \mid \#_a(w) = 0 \mod (3)\}$



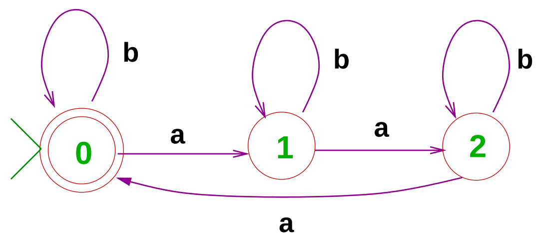
and

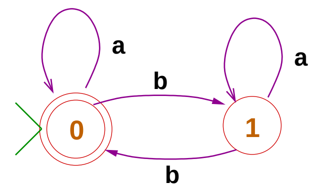
- M_2 recognizes $L_2 = \{w \in \Sigma^* \mid \#_b(w) = 0 \mod (2)\}$.

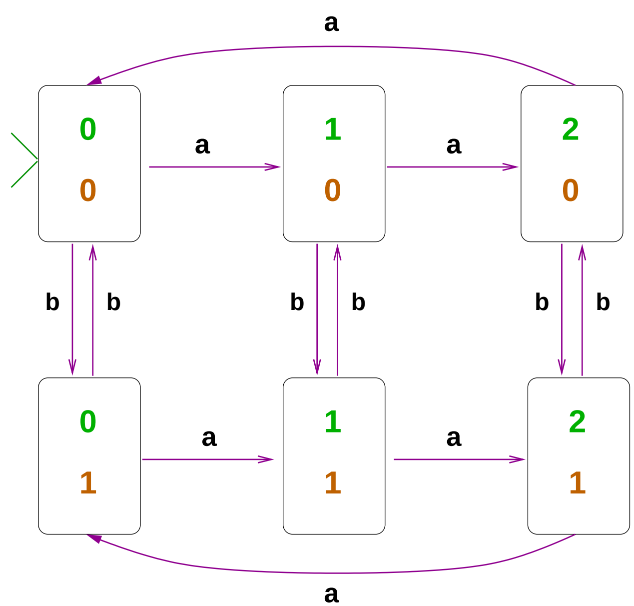


$\#_b w \;=\; 0 \mod 2$

Two automata collaborating

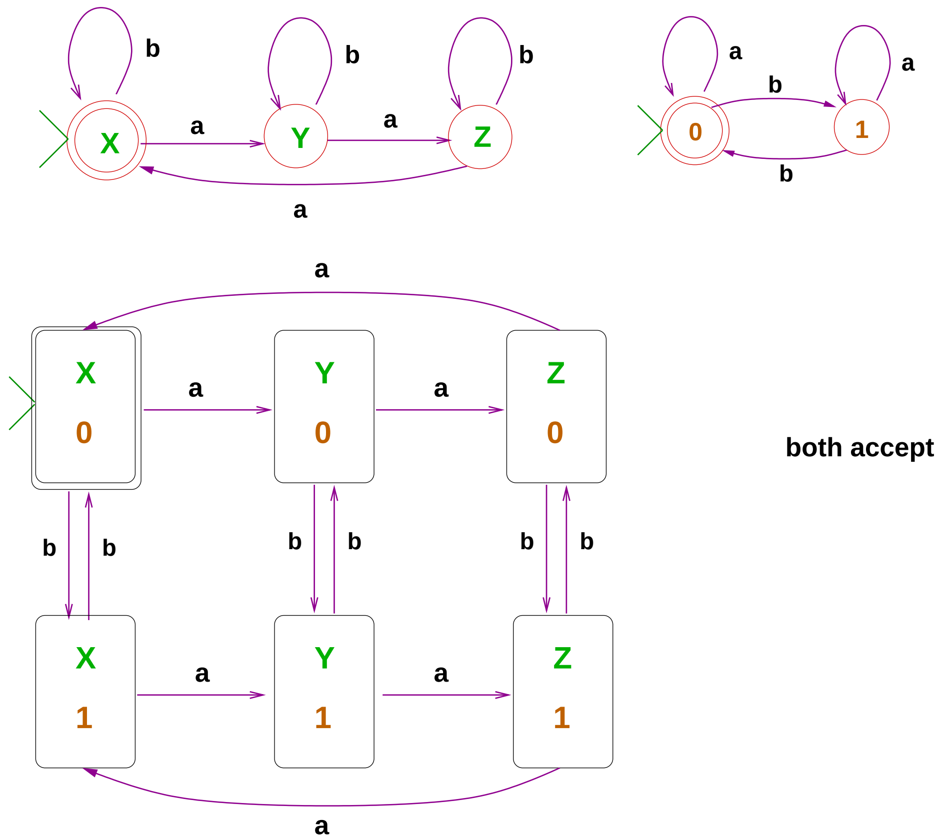






Conjuctive pairing

- Accepting when both accept:



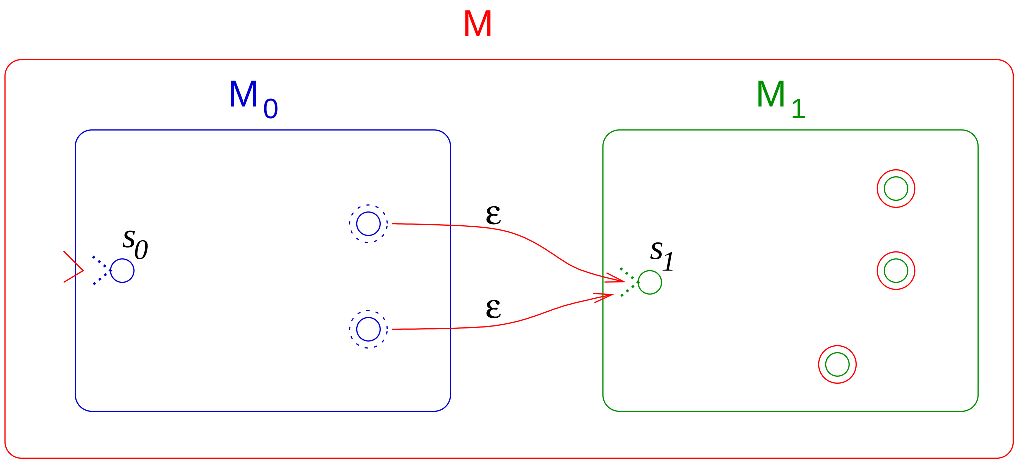
both accept

The concatenation of recognized languages is recognized

- Given $L_0 = \mathcal{L}(M_0)$ where $M_0 = (Q_0, s_0, A_0, \delta_0)$ and $L_1 = \mathcal{L}(M_1)$ where $M_1 = (Q_1, s_1, A_1, \delta_1)$.

The concatenation of recognized languages is recognized

- Given $L_0 = \mathcal{L}(M_0)$ where $M_0 = (Q_0, s_0, A_0, \delta_0)$ and $L_1 = \mathcal{L}(M_1)$ where $M_1 = (Q_1, s_1, A_1, \delta_1)$.
- Here's an NFA M that recognizes $L_0 \cdot L_1$:



The concatenation of recognized languages is recognized

- Given $L_0 = \mathcal{L}(M_0)$ where $M_0 = (Q_0, s_0, A_0, \delta_0)$ and $L_1 = \mathcal{L}(M_1)$ where $M_1 = (Q_1, s_1, A_1, \delta_1)$.
- If $w = u \cdot v$ where $u \in L_0$ and $v \in L_1$
  then $s_0 \xrightarrow{u} a_0 \xrightarrow{\epsilon} s_1 \xrightarrow{v} a_1$
  for some $a_0 \in A_0$ and $a_1 \in A_1$,
  M accepts w .

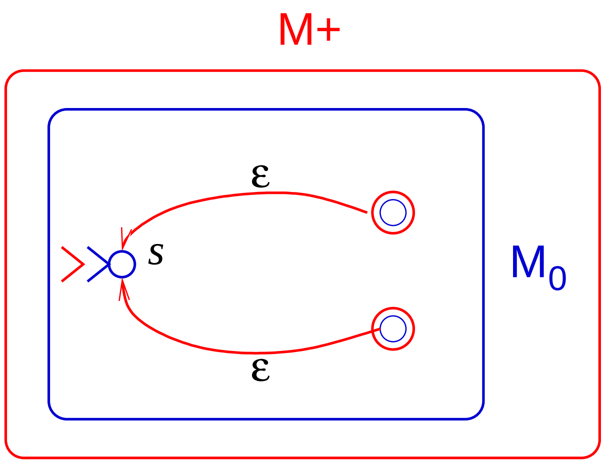
The concatenation of recognized languages is recognized

- Given $L_0 = \mathcal{L}(M_0)$ where $M_0 = (Q_0, s_0, A_0, \delta_0)$ and $L_1 = \mathcal{L}(M_1)$ where $M_1 = (Q_1, s_1, A_1, \delta_1)$.

- Conversely, Suppose w is accepted by M, $s_0 \xrightarrow{w} A_1$.
  The trace starts in Q_0 and ends in Q_1,
  so it must have a transition $q \to p$ for some $q \in Q_0$ and $p \in Q_1$.
  The only such transitions are $a \xrightarrow{\epsilon} s_1$ for $a \in A_0$. M has no trasitions from Q_1 to Q_0, so the trace must be for $s_0 \xrightarrow{u} a \xrightarrow{\epsilon} s_1 \xrightarrow{v} a'$ for some u accepted by M_0 and some v accepted by M_1. Hence $w = u \cdot v \in L_0 \cdot L_1$.

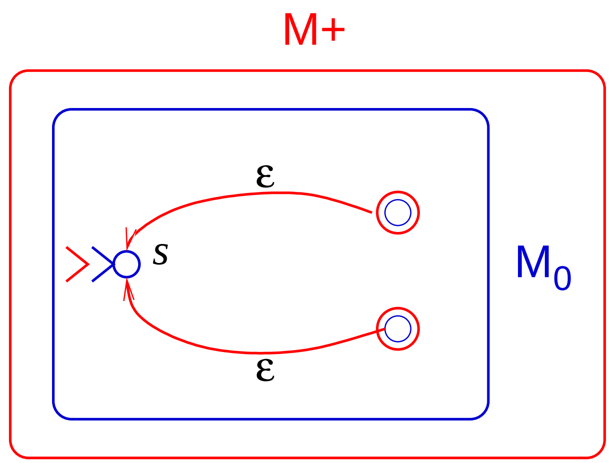
The plus and star of a recognized language are recognized

- Given a language $L = \mathcal{L}(M)$ here's an NFA M^+ recognizing L^+:



The plus and star of a recognized language are recognized

- Given a language $L = \mathcal{L}(M)$ here's an NFA M^+ recognizing L^+:



- Since $L^* = L^+ \cup \{\varepsilon\}$, L^* is also recognized.

Every basic language is recognized

- Induction on the dfn of basic languages. We showed:

Every basic language is recognized

- Induction on the dfn of basic languages. We showed:
- Finite languages are recognized.

Every basic language is recognized

- Induction on the dfn of basic languages. We showed:
- Finite languages are recognized.
- Set operations yield recognized languages from recognized languages (proofs using DFAs)

Every basic language is recognized

- Induction on the dfn of basic languages. We showed:
- Finite languages are recognized.
- Set operations yield recognized languages from recognized languages (proofs using DFAs)
- Language operations yield recognized languages from recognized languages (proofs using NFAs)

Every basic language is recognized

- Induction on the dfn of basic languages. We showed:
- Finite languages are recognized.
- Set operations yield recognized languages from recognized languages (proofs using DFAs)
- Language operations yield recognized languages from recognized languages (proofs using NFAs)
- So by induction on basic language every basic language is recognized.

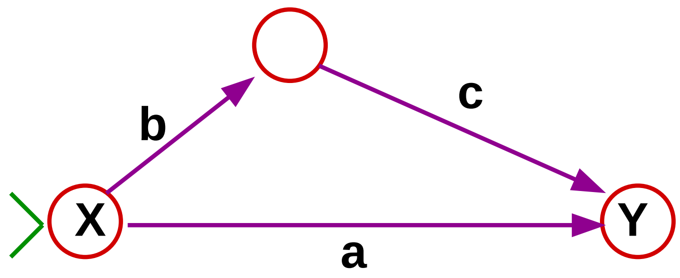
Uniting three definitions (reminder)

- We'll see that the following properties of languages are equivalent.
  - L is basic
  - L is recognized by an automaton IMPLIES
  - L is regular
  - L has finitely many residues

EVERY RECOGNIZED LANGUAGE IS REGULAR

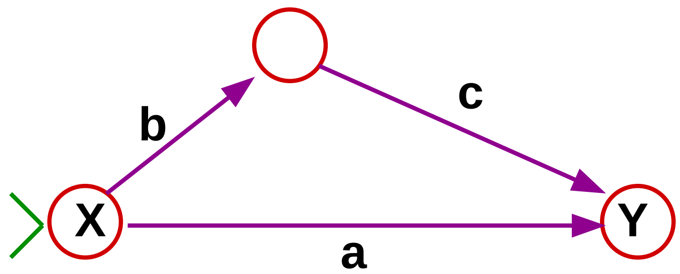
Getting from here to there

- What strings are leading from **X** to **Y**?



Getting from here to there

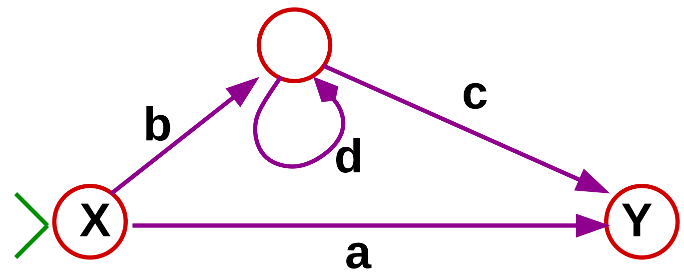
- What strings are leading from **X** to **Y**?



a ∪ bc

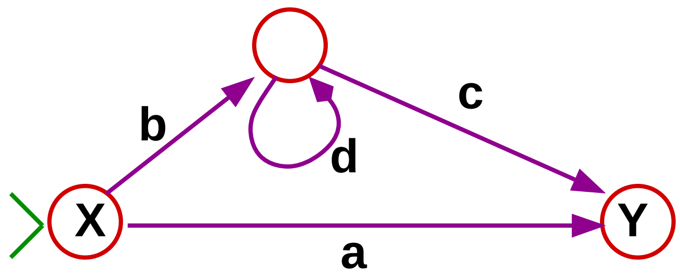
Getting from here to there

- What strings are leading from **X** to **Y**?



Getting from here to there

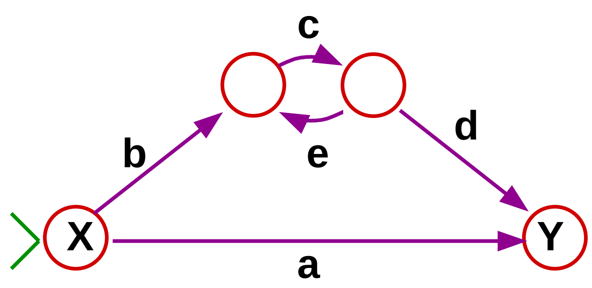
- What strings are leading from **X** to **Y**?



$a \cup bd^*c$

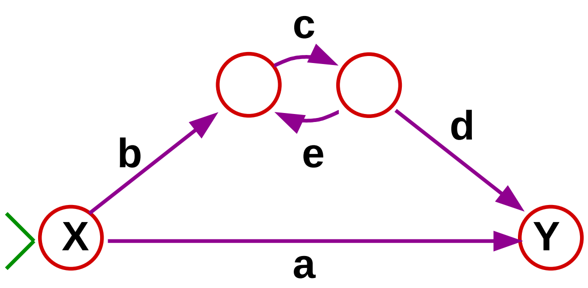
Getting from here to there

- What strings are leading from **X** to **Y**?



Getting from here to there

- What strings are leading from **X** to **Y**?

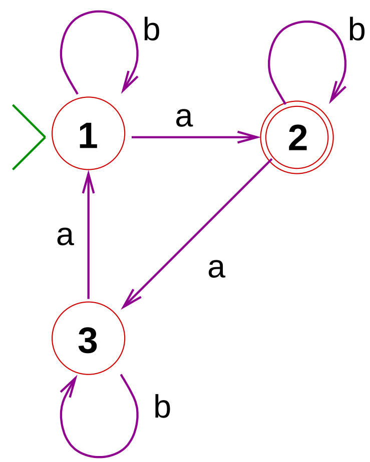


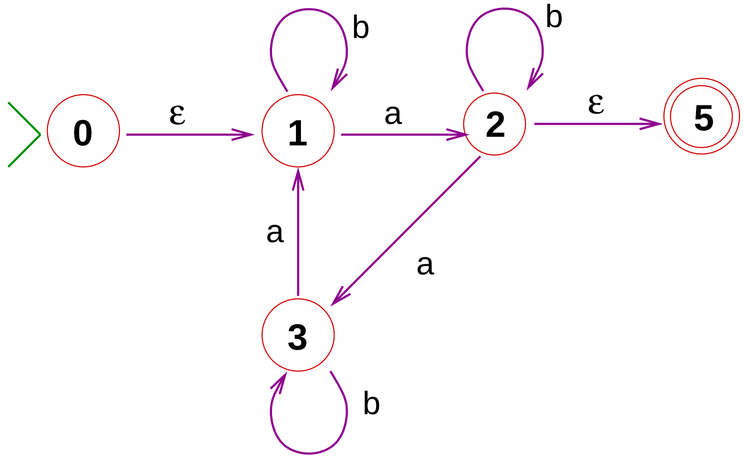
a ∪ b(ce)*cd

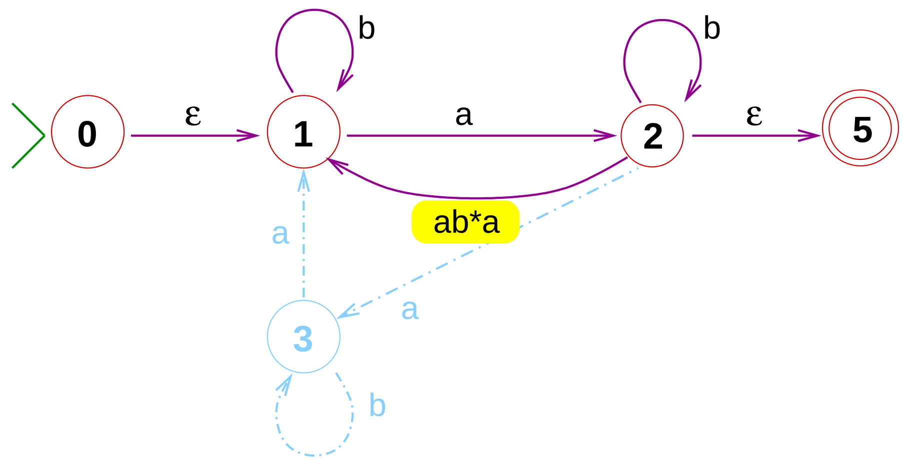
Graphs with reg-exps as labels

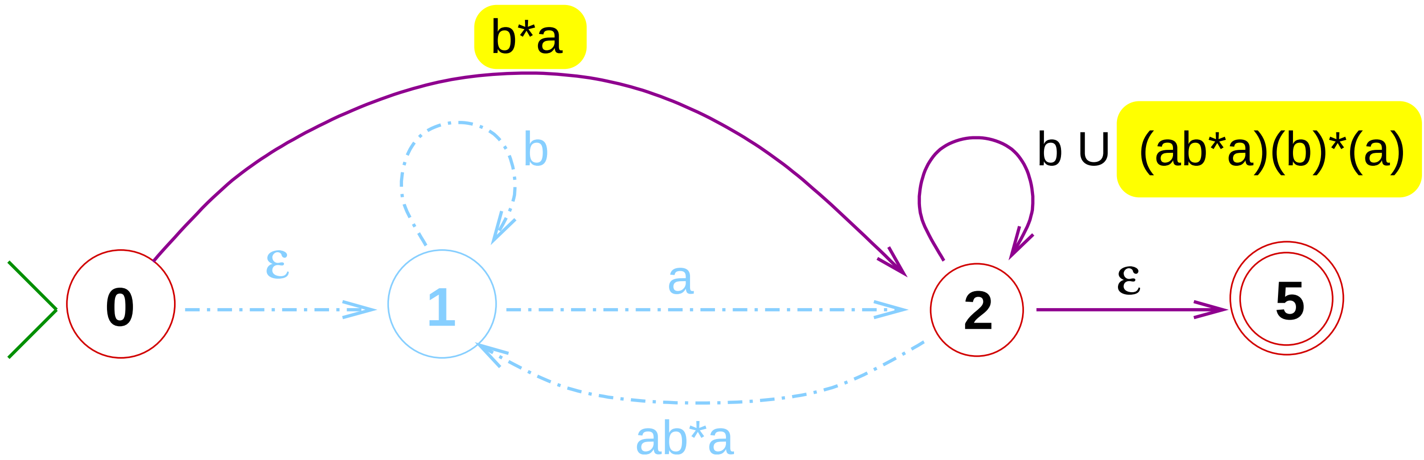
- Starting with the given NFA,
  Collapse labels: eg, replace $q \xrightarrow{a,b,\epsilon} p$ by $q \xrightarrow{a \cup b \cup \epsilon} p$
- Create a new start state s_0
with an ε-transition to the original start state of N.
- Create a new state a_0 as the only accepting state,
  and create an ε-transition from each accepting state of N to a_0.

A working example

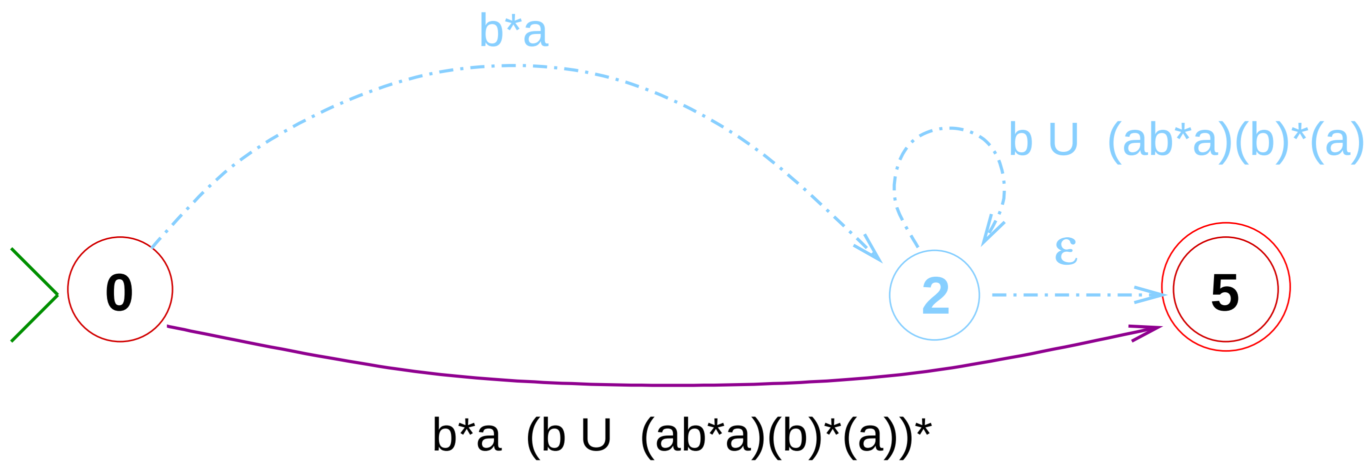



b
b
0
ε
1
a
2
ε
5
a
a
3
b


b
b
0
ε
1
a
2
ε
5
ab*a
a
a
3
b

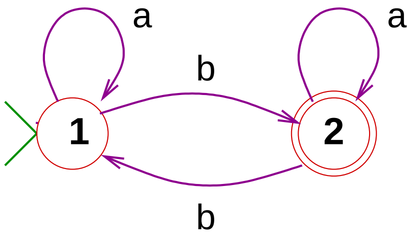


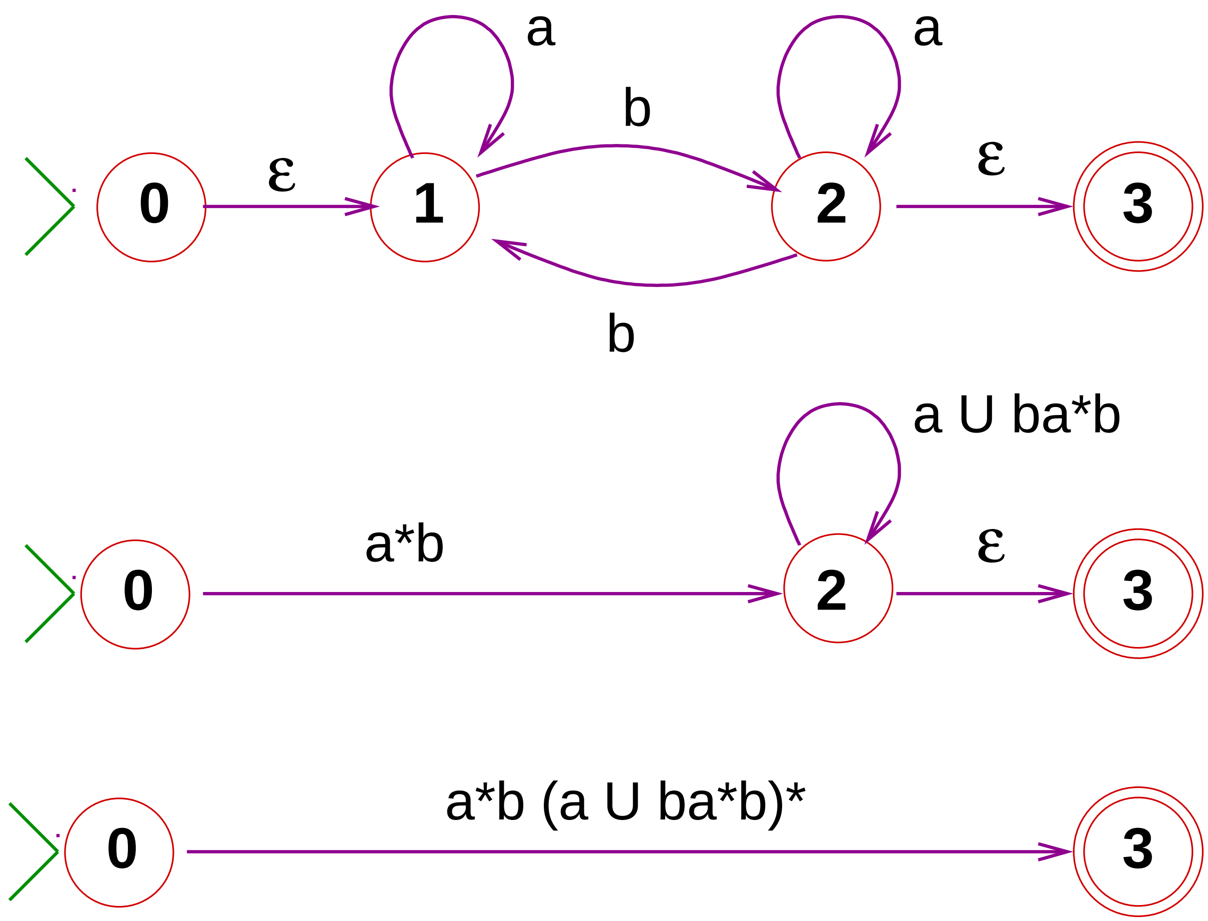
b*a
b
b U (ab*a)(b)*(a)
0
ε
1
a
2
ε
5
ab*a



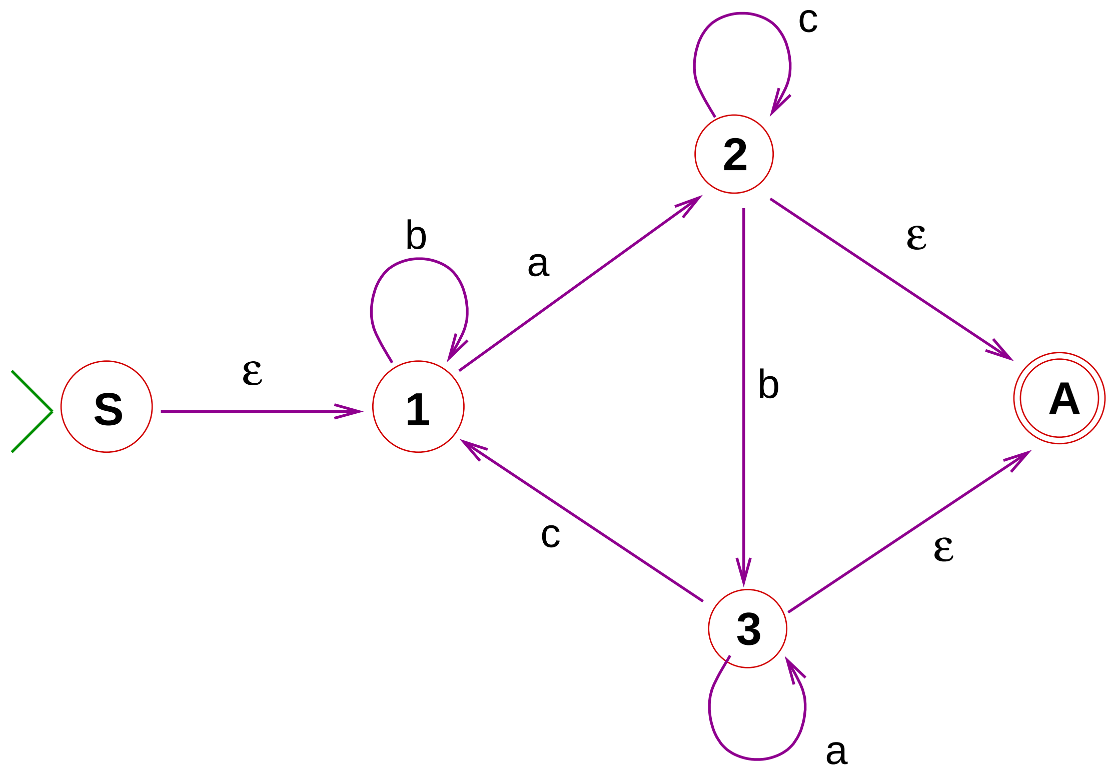
$$\mathcal{L}(N) = \mathcal{L}(\mathtt{b}^{*}\cdot\mathtt{a}\cdot(\mathtt{b}\cup(\mathtt{a}\cdot\mathtt{b}^{*}\cdot\mathtt{a})\cdot(\mathtt{b})^{*}\cdot(\mathtt{a}))^{*})$$

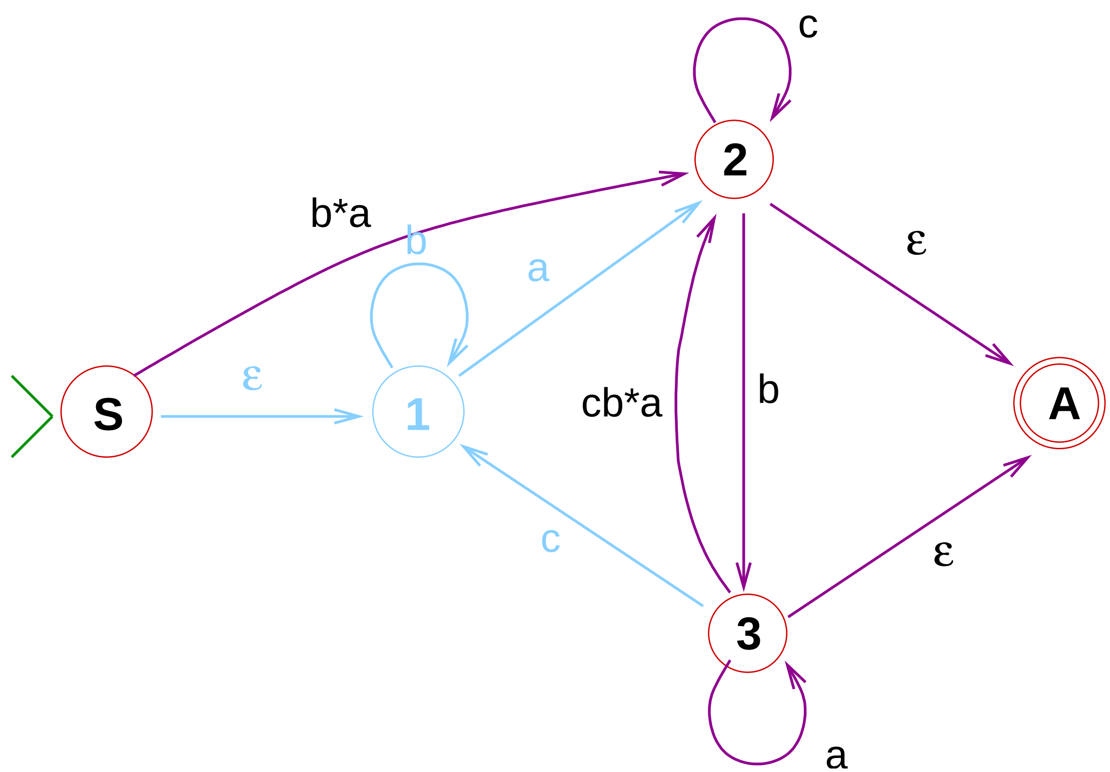
Another example

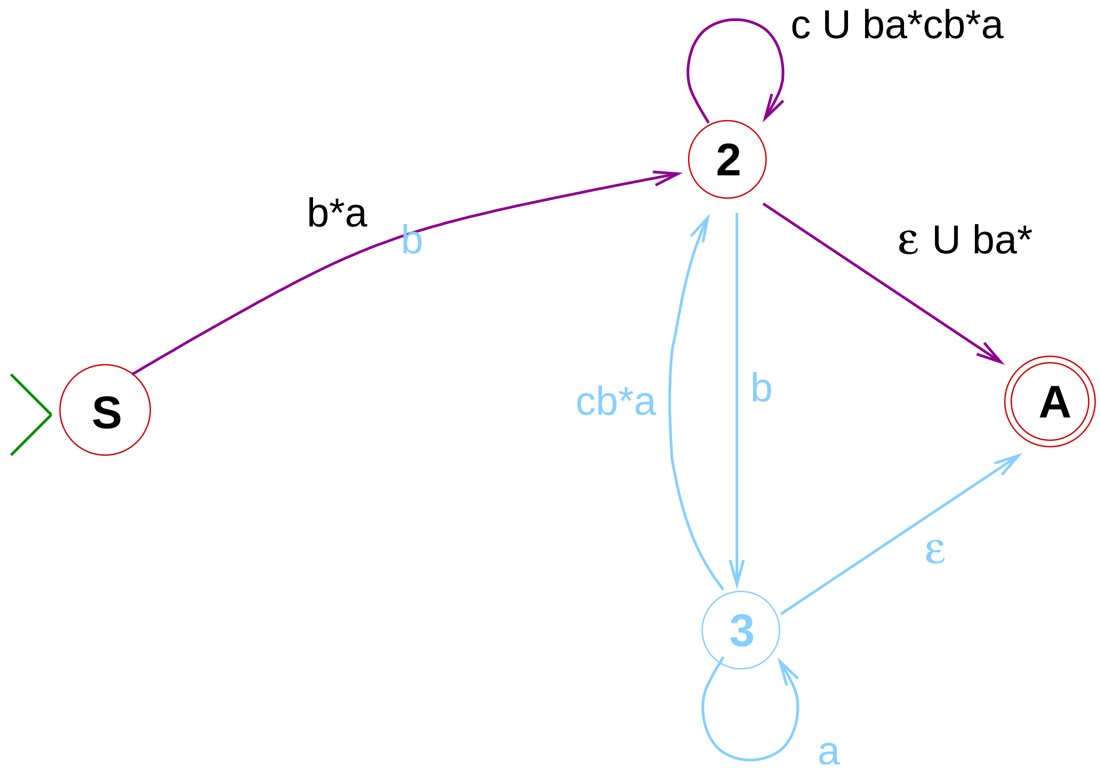


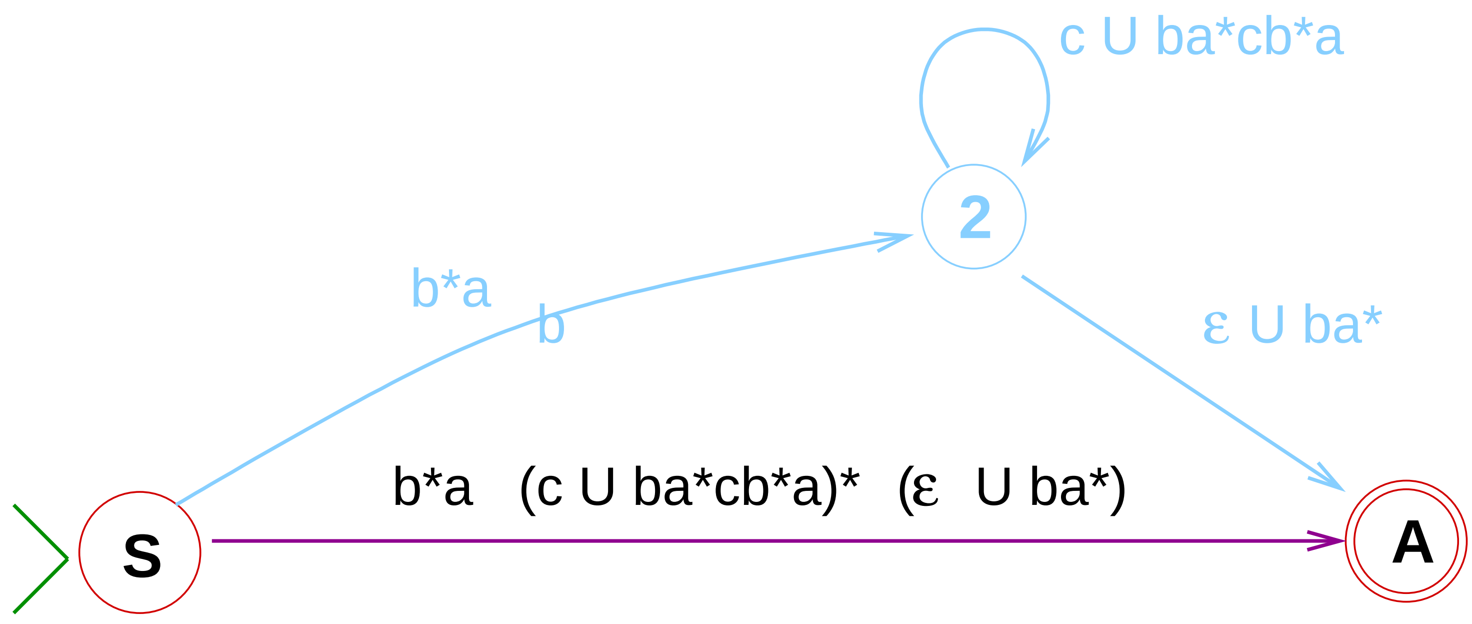

a
a
b
ε
0
ε
1
2
3
b
a U ba*b
a*b
ε
0
2
3
a*b (a U ba*b)*
0
3

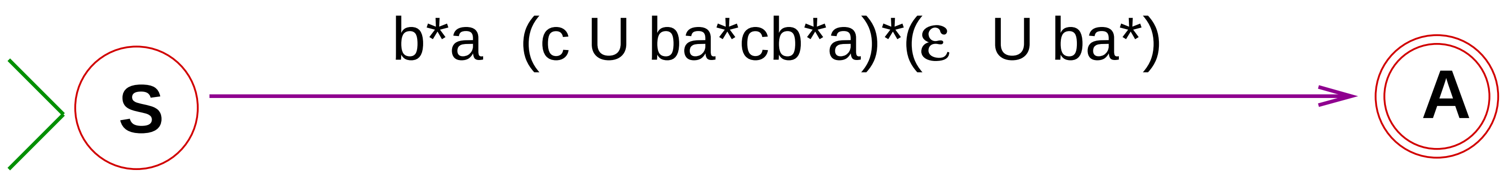
Yet another example




c
2
b*a
b
a
ε
S
ε
1
cb*a
b
A
c
ε
3
a


c U ba*cb*a
2
b*a
b
ε U ba*
cb*a
b
S
A
ε
3
a


c U ba*cb*a
2
b*a
b
ε U ba*
b*a (c U ba*cb*a)* (ε U ba*)
S
A


b*a (c U ba*cb*a)*(ε U ba*)
S
A

The underlying math

- NFAs are not generated from components:
  transition rules can go any which way.
- So how can be reason inductively about all NFAs?
- Look closer to what we want to prove:
  Given an NFA $M = (Q, s, A, \Delta)$ over an alphabet Σ,
  find a regular expression that denotes $\{w \mid s \xrightarrow{w} A\}$.
- As was the case for the Unique Parsing Theorem,
  we up the ante to make this work.

Limitting the stepping stones

- For sets $T \subseteq Q$, consider the relation $q \xrightarrow{w(T)} p$ that holds when
  w leads from q to p using only states in T.
- In particular $q \xrightarrow{w(Q)} p$ means $q \xrightarrow{w} p$.
- Goal: For states q, p and $T \subseteq Q$ $\{w \mid q \xrightarrow{w(T)} p\}$ is denoted by some regexp $\alpha^{q \to p(T)}$.
- Base: $T = \emptyset$, and $\alpha^{q \to p(\emptyset)}$ must denote the set of $\sigma \in \Sigma_\epsilon$
  for which $q \xrightarrow{\sigma} p$ is in the transition.
  Take the union of those.
- Step: Given T and state $r \notin T$,
  and considering $T \cup \{r\}$,
  define $\alpha^{q \to p}_{T+r}$ in terms of expressions $\alpha^{\cdots}_{T}$.
- We have $q \xrightarrow{w(T+r)} p$ iff either $q \xrightarrow{w(T)} p$ or $w = u \cdot x_1 \cdot \cdots \cdot x_k \cdot v$,
  where $q \xrightarrow{u(T)} r \xrightarrow{x_1(T)} r \cdots \xrightarrow{x_k(T)} r \xrightarrow{v(T)} p$

- So define $\alpha_{T+r}^{q \to p} = \alpha_T^{q \to p} \cup \alpha_T^{q \to r} \cdot (\alpha_T^{r \to r})^* \cdot \alpha_T^{r \to p}$
- One concern: to preserve info about acceptance we should not eliminate the start state or any accepting state.
- Solution:

  1. New start s_0 with $s_0 \xrightarrow{\epsilon} s$;
  2. New unique accept a_0 with $a \xrightarrow{\epsilon} a_0$ for each $a \in A$.
  3. Now $\mathcal{L}(N) = \alpha_Q^{s_0 \to a_0}$. QED

- We showed an algorithmic implementation of the construction above.

TWO-WAY DFAs

A stronger read-only deterministic device

- Consider the language L over $[\mathtt{a}-\mathtt{z}]$
  of words that include all letters.
  No English word is in L, but probably every book.
- L is a regular language: it is the intersection
  of the 26 languages $\{w \mid w$ has $\sigma\}$ for $\sigma = \mathtt{a}, \mathtt{b}....$
- The smallest DFA that recognizes L
  has $> 2^{26} > 67,000,000$ states.
- The smallest NFA recognizing L has 27 states.
- Is there a *deterministic* *algorithm*
  that does it with a manageable number of states?

A deterministic algorithm for the all-letters problem

- Algorithm: Scan for each digit separately, and repeat.

- This cannot be done if we only read forward!
  The cursor would have to be scrolled back (or repositioned).
- SO let’s imagine a device that behaves just like an automaton,
  but can move the cursor both ways.

Some challenges

- Symbol read determines not only next state,
  but also next move: forward or backward.
- To detect the ends of the input string it must have end-markers,
  say $>$ (the ***gate***) on the left,
  and $\sqcup$ (the ***blank***) on the right.
- Termination is not by reading through,
  but needs to be declared by a final accept state.
  (We need not guarantee termination.)

Two-way automata

A ***two-way automaton (2DFA)*** over an alphabet Σ:

- Finite set of states Q
- $s \in Q$, the *initial state*
- $a \in S$, the *accepting state*
- Transition partial-function: $\delta : Q \times \Gamma \rightharpoonup Q \times \mathbf{Act}$
  where $\Gamma = \Sigma \cup \{>, \sqcup\}$ and $\mathbf{Act} = \{+, -\}$.
- Write $q \overset{\sigma(\alpha)}{\to} p$ for $\delta(q, \sigma) = \langle p, \alpha \rangle$

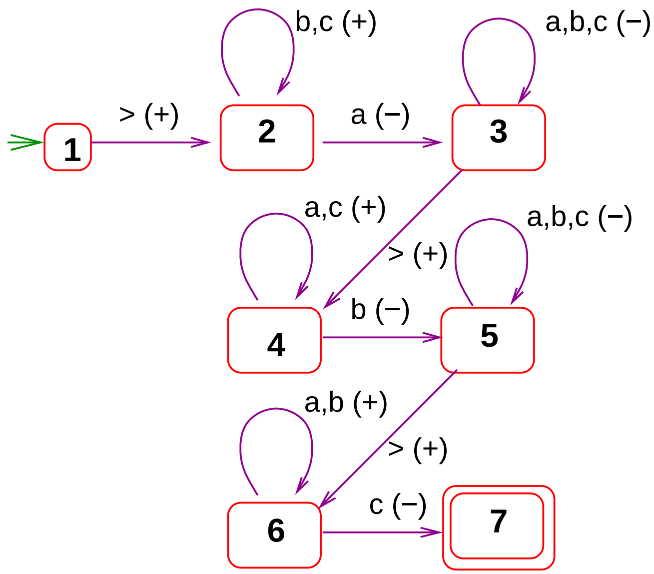
Two-way automata

- $\delta : \; Q \times \Gamma \rightharpoonup Q \times \mathbf{Act}$
  where $\Gamma = \Sigma \cup \{>, \sqcup\}$ and $\mathbf{Act} = \{+, -\}$.
- Write $q \overset{\sigma(\alpha)}{\rightarrow} p$ for $\delta(q, \sigma) = \langle p, \alpha \rangle$

The intent:

- Γ end-markers $>$ ***(gate)*** and $\sqcup$ ***(blank)*** added to Σ
- Example: Input **001201** appears as $>$**001201**$\sqcup$
- The ***actions*** $+$ and $-$ stand for "step forward" and "step back."

Example: The strings using all of `a`,`b`,`c`



- With 26 in place of 3 we'd have 53 states,
  as opposed to $> 67,000,000$ states in the smallest DFA!

Operation of 2DFAs: configurations

- For DFAs we could generate the relation $p \xrightarrow{w} q$ inductively, as a function of w.
- This is no longer the case for 2DFAs: here we *must* account for the cursor position and keep record of the entire input for future use.
- A ***cursored-string*** over Σ is a $\Sigma-$string with one underlined symbol-position.
- A ***configuration (cfg)*** is a pair $(q, \check{w})$ where
  - q is a state, and
  - $\check{w}$ is a cursored-string,

  That is, (state, cursored-string).
- Example: $(q, >0101\underline{1}00\sqcup)$
- The ***initial cfg for input w*** is the cfg $(s, \underline{>}w\sqcup)$.

The YIELD relation

- The **Yield** relation $\Rightarrow$
  (or $\Rightarrow_M$ when it matters which M) is obtained by:
-
  - If $q \overset{\gamma(+)}{\rightarrow} p$
    then $(q, u\underline{\gamma}\tau v) \Rightarrow (p, u\gamma\underline{\tau}v)$
  - If $q \overset{\gamma(-)}{\rightarrow} p$
    then $(q, u\tau\underline{\gamma}v) \Rightarrow (p, u\underline{\tau}\gamma v)$
  - Nothing else
- If the given cfg is $(q, 01101\underline{0})$,
  and $q \overset{0(+)}{\rightarrow} p$, then the transition above does not apply.

  The same holds when invoking a transition $q \overset{0(-)}{\rightarrow} p$
  for a configuration with a cursor at the head of the string, such as $(q, \underline{0}11010)$.

Traces, acceptance, recognition

- A cfg $c = (q, u\gamma v)$ is ***terminal*** if no transition applies (no yield). It is a ***accepting*** its state is accepting state a.
- A ***trace*** of M for input w is a sequence of

$$c_0 \Rightarrow c_1 \Rightarrow \cdots$$

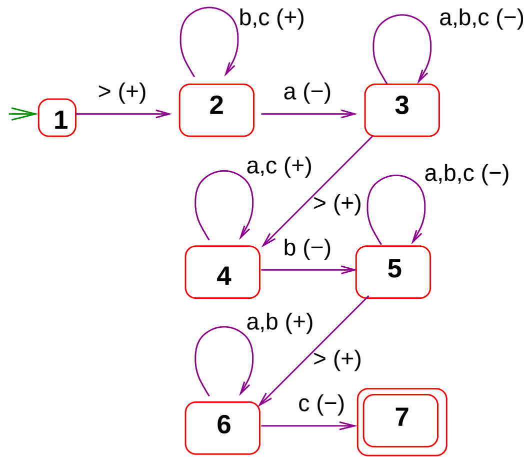
  where c_0 is initial for w, and either

  1. the sequence is infinite; or
  2. the sequence is finite, and its last cfg is terminal.

- The trace is ***accepting*** if it is finite and its last cfg is accepting.
- M ***accepts*** $w \in \Sigma^*$ if it its trace for input w is accepting.
- The language ***recognized*** by M is

$$\mathcal{L}(M) = \{w \in \Sigma^* \mid M \text{ accepts } w\}$$

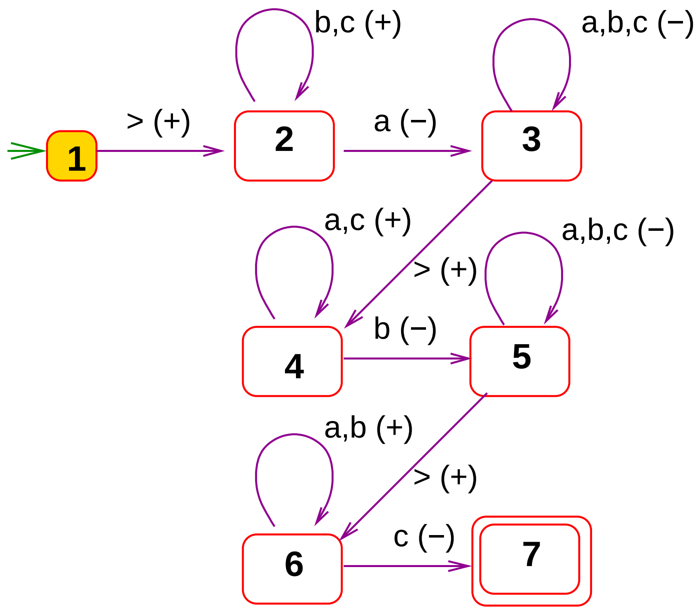
Example



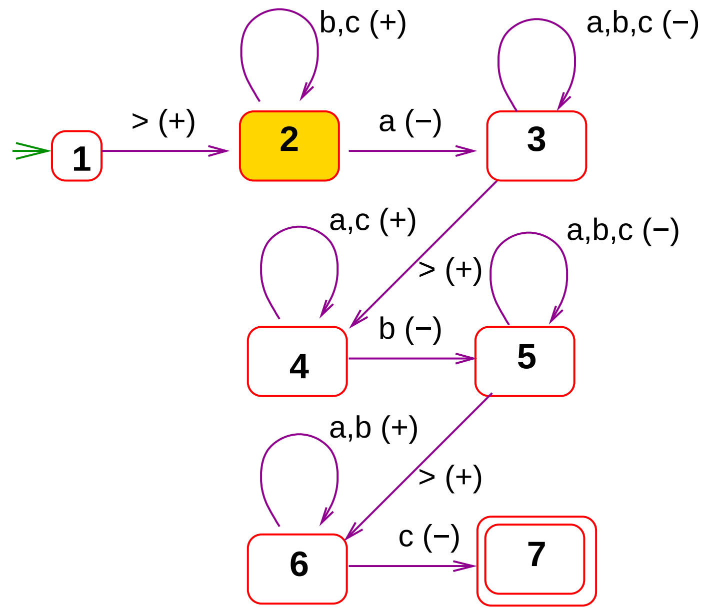
Accepting trace for trace of M above for $w = \mathbf{bcab}$:

$$
\begin{array}{ll}
(1, \underline{>}\mathbf{bcab}\sqcup) & \\
\Rightarrow (2, >\underline{\mathrm{b}}\mathrm{cab}\sqcup) & \Rightarrow (4, >\underline{\mathrm{b}}\mathrm{cab}\sqcup) \\
\Rightarrow (2, >\mathrm{b}\underline{\mathrm{c}}\mathrm{ab}\sqcup) & \Rightarrow (5, \underline{>}\mathrm{bcab}\sqcup) \\
\Rightarrow (2, >\mathrm{bc}\underline{\mathrm{a}}\mathrm{b}\sqcup) & \Rightarrow (6, >\underline{\mathrm{b}}\mathrm{cab}\sqcup) \\
\Rightarrow (3, >\mathrm{b}\underline{\mathrm{c}}\mathrm{ab}\sqcup) & \Rightarrow (6, >\mathrm{b}\underline{\mathrm{c}}\mathrm{ab}\sqcup) \\
\Rightarrow (3, >\underline{\mathrm{b}}\mathrm{cab}\sqcup) & \Rightarrow (7, >\underline{\mathrm{b}}\mathrm{cab}\sqcup) \\
\Rightarrow (3, \underline{>}\mathrm{bcab}\sqcup) &
\end{array}
$$

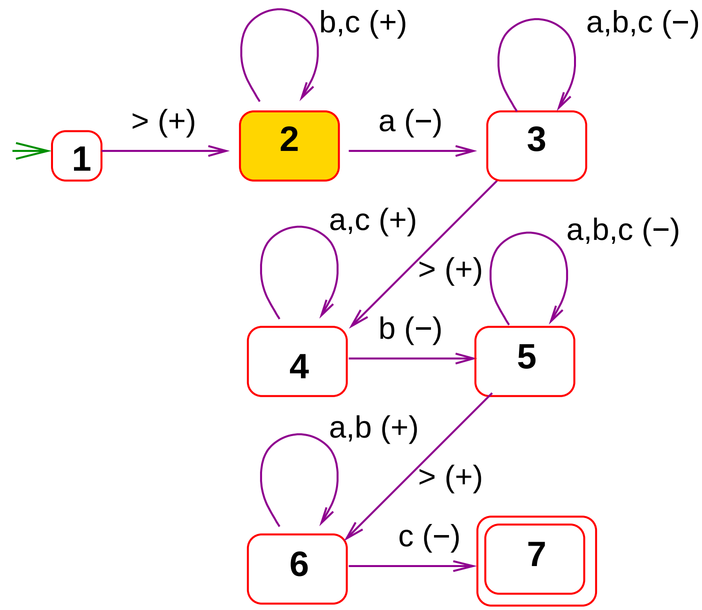
$(1, \geq \mathbf{bcab} \sqcup)$



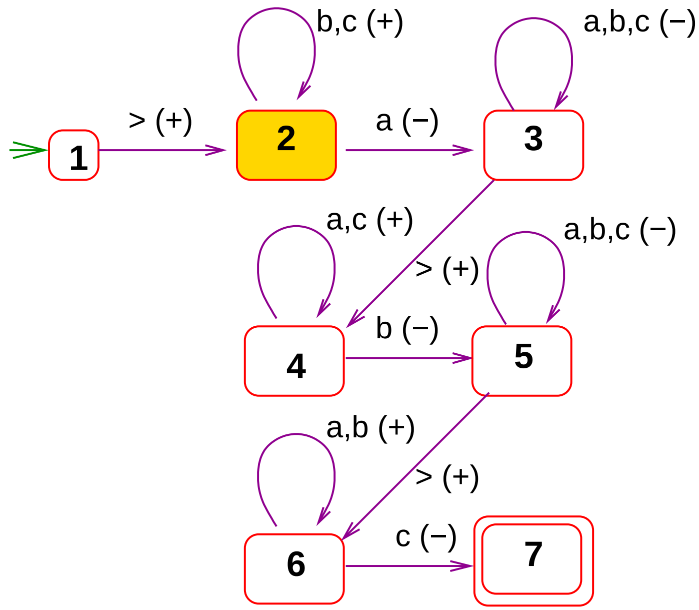
$(2, >\underline{\mathtt{b}}\mathtt{cab}\sqcup)$



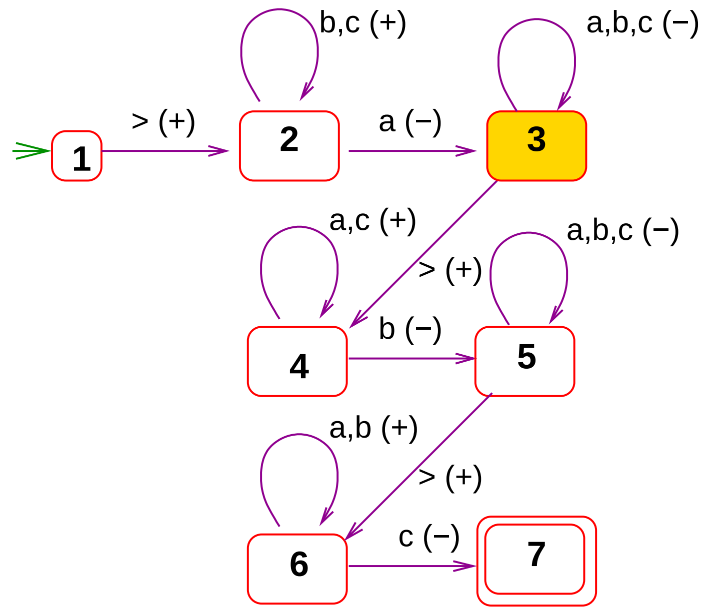
$(2, >\mathrm{b}\underline{\mathrm{c}}\mathrm{ab}\sqcup)$



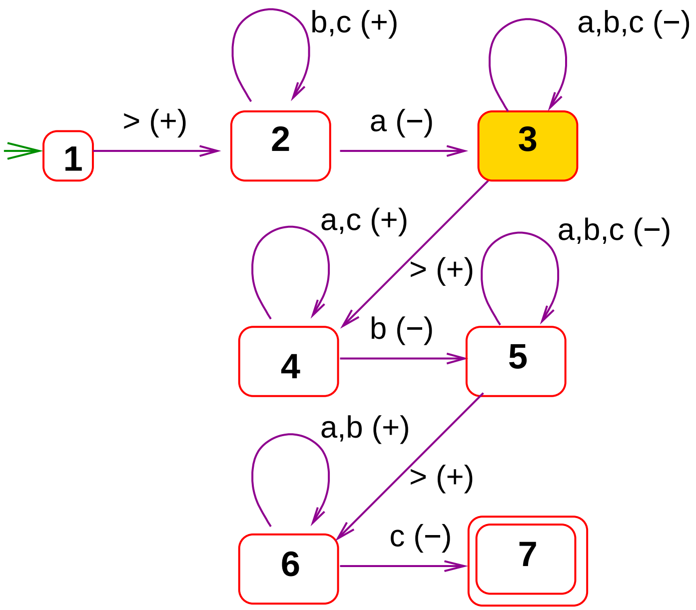
$(2, >\mathtt{bc}\underline{\mathtt{a}}\mathtt{b}\sqcup)$



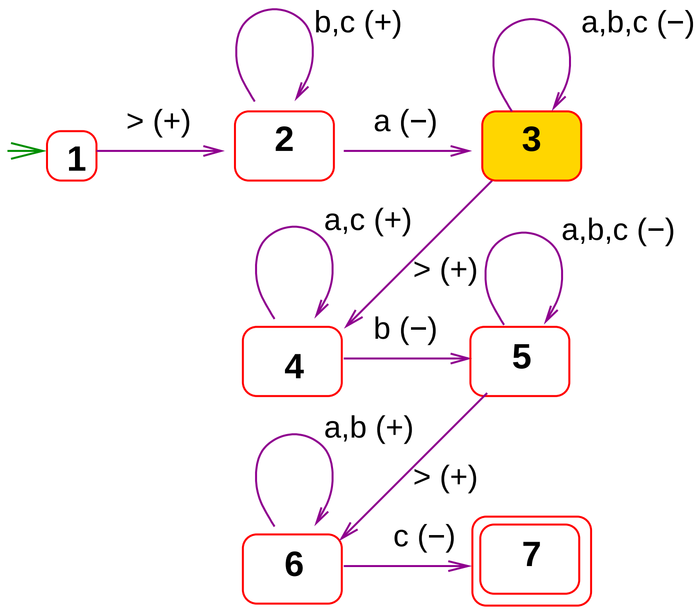
$(3,\ >b\underline{c}ab\sqcup)$



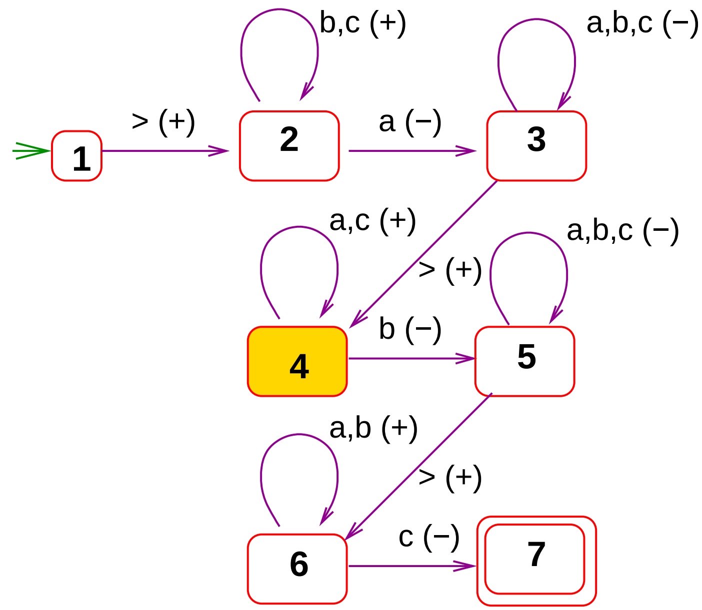
(3, >bcab⊔)



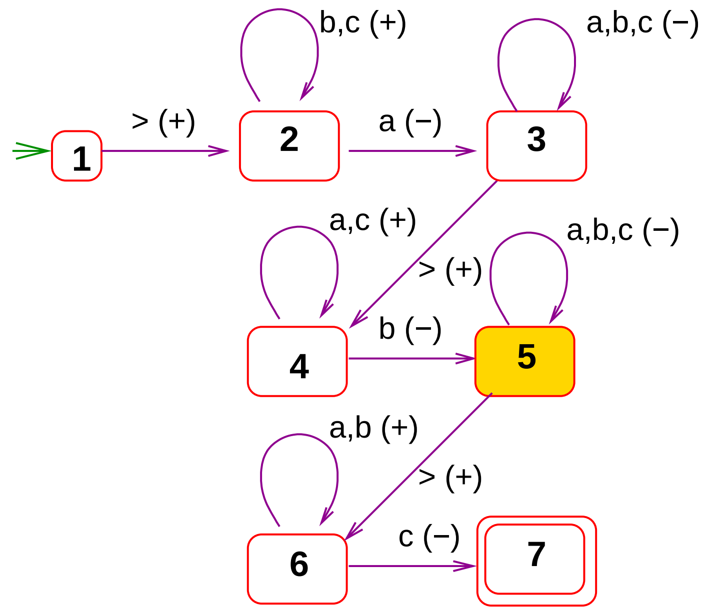
$(3, \underline{>}\mathtt{bcab}\sqcup)$



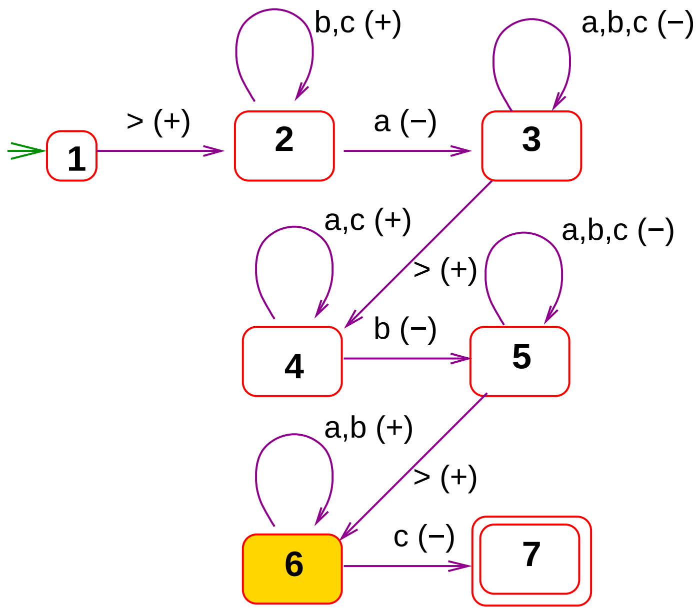
$(4, >\underline{b}cab\sqcup)$



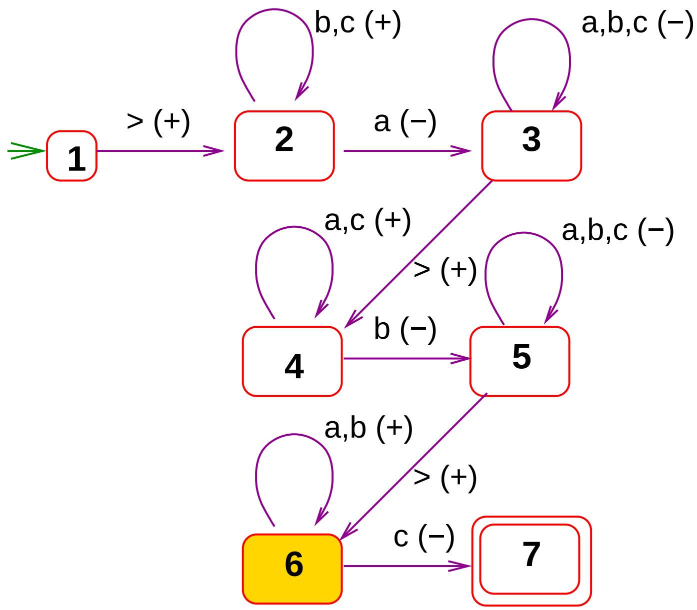
$(5, \underline{>}\text{bcab}\sqcup)$



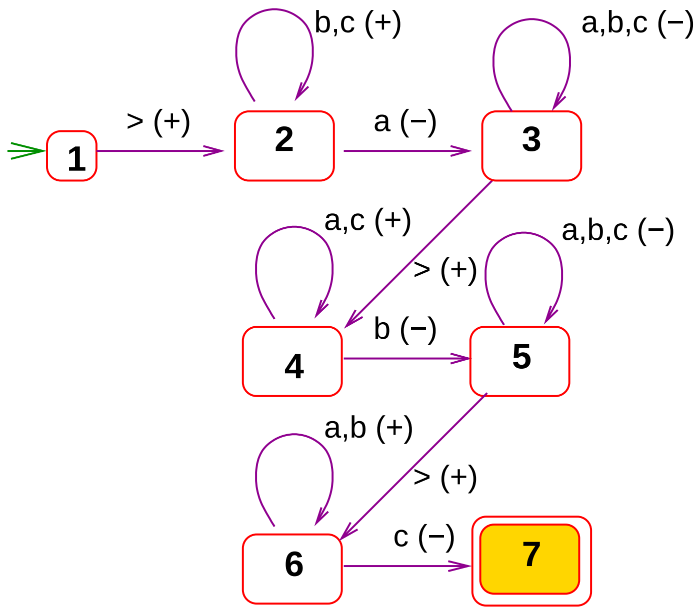
$(6, \triangleright \underline{b}cab\sqcup)$



$(6, >b\underline{c}ab\sqcup)$



$(7, >\underline{b}cab\sqcup)$

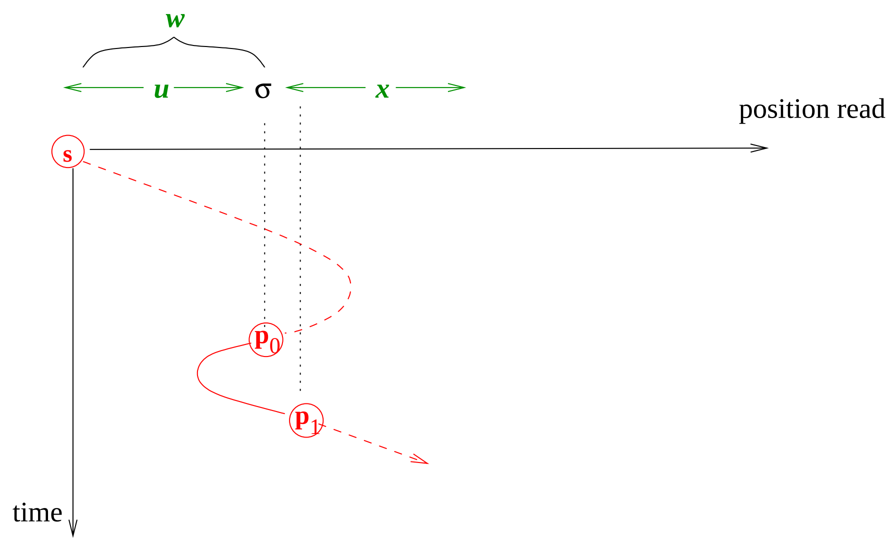


Two-way automata recognize just regular languages!

- Yet another characterization of regular languages!
- Adding nondeterminism to 2DFA still recognizes just regular languages!
- We still avoid extensible memory, so this is not a big surprise.

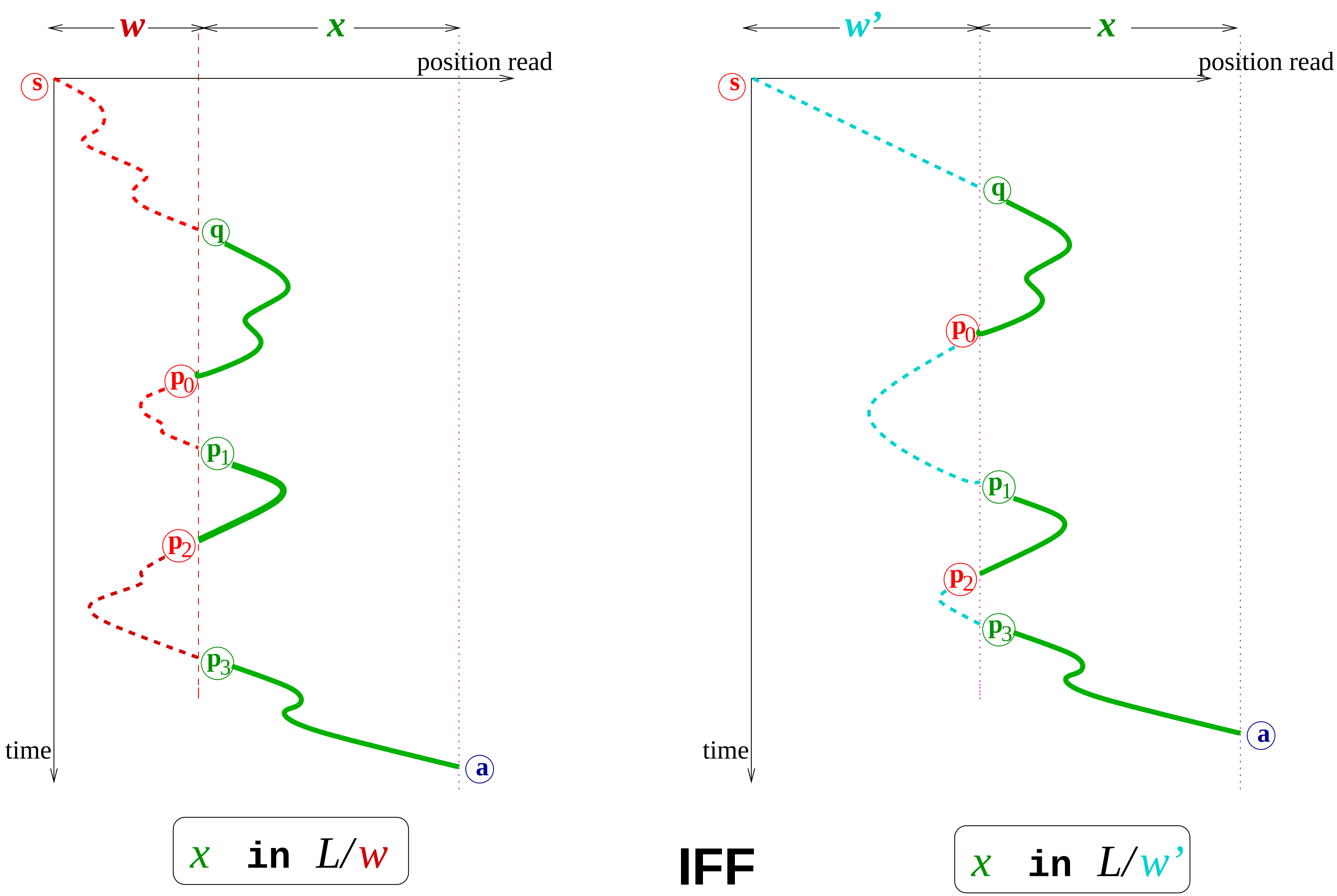
Proof outline

- DFA recognize languages with finitely many residues L/w.
- For each w a finite amount of info suffices to decide $x \in L/w$.
- For DFA the info is the state q reached: $s \xrightarrow{w} q$.
- For 2DFA the scan might cross out of w and into x .
  back in, and then out again into x.
- This is the info needed about w:
  If the reading cross back into w in a state
- The extra info:
  the pairs $(\textit{in}, \textit{out})$ of states
  s.t. crossing back into w in state *i*n
  leads to crossing back out in state *o*ut.


w
u
σ
x
position read
s
p0
p1
time

Every language recognized by a is regular!

- Say that $\langle p_0, p_1 \rangle$ is a *back-crossing pair.*
- L/w <u>is</u> determined by q reached by reading w,
  plus the set of back-crossing pairs for w:
  if w, w' reach the same state,
  and have the same crossing pairs, then $L/w = L/w'$.



w
x
position read
s
q
p0
p1
p2
p3
a
time
x in L/w
w'
x
position read
s
q
p0
p1
p2
p3
a
time
x in L/w'
IFF

- For M with k states
  there are k^2 potential back-crossing pairs,
  and so 2^{k^2} possible descriptions of the situation at the border.
- Finitely many residues, albeit a lot, but still
  recognizing a regular language!