Fall 2022

Strings and languages

Aug 2022

1. For each of the following inequalities give an example of languages L for which the inequality holds.

   (a) $(L^2)^* \neq (L^*)^2$. (Hint: One letter suffice)

   (b) $L \cdot L^* \neq L^*$.

   Solution. Let $L = \{\mathtt{a}\}$. Then $\varepsilon \in L^*$ but $\varepsilon \notin L \cdot L^*$.

2. For each of the following inequalities give an example of languages L, M for which the inequality holds.

   (a) $L \cdot M \neq M \cdot L$.

   Solution. Let $L = \{\mathtt{0}\}$ and $M = \{\mathtt{1}\}$. Then $L \cdot M = \{\mathtt{01}\} \neq \{\mathtt{10}\} = M \cdot L$.

   (b) $L^+ \cdot L^+ \neq L^+$.

   Solution. Let $L = \{\mathtt{1}\}$. Then $\mathtt{1} \in L^+$ but $\mathtt{1} \notin L^+ \cdot L^+$ since all strings in $L^+ \cdot L^+$ have length $\geqslant 2$.

   (c) $(L \cdot M)^* \neq L^* \cdot M^*$.

   Solution. Let $L = \{\mathtt{0}\}$ and $M = \{\mathtt{1}\}$. Then $\mathtt{0101} = \mathtt{01} \cdot \mathtt{01} \in (L \cdot M)^*$ but $\mathtt{0101} \notin L^* \cdot M^*$ since no string in $L^* \cdot M^*$ can have $\mathtt{1}$ preceding $\mathtt{0}$.

   An even simpler example: $(\emptyset \cdot \{\mathtt{1}\})^* = \emptyset^* = \{\varepsilon\}$ but $\emptyset^* \cdot \{\mathtt{1}\}^* = \{\varepsilon\} \cdot \{\mathtt{1}\}^* = \{\mathtt{1}\}^*$.

3. Prove that $L^* \cdot L^* = L^*$ for all languages L.

   Solution. We have $\varepsilon \in L^*$ so $L^* \cdot L^* \supseteq \{\varepsilon\} \cdot L^* = L^*$.

   To prove that $L^* \cdot L^* \subseteq L^*$ suppose $w \in L^* \cdot L^*$, that is $w = u \cdot v$ for some $u, v \in L^*$. By the definition of L^* we have then $u \in L^i$ and $v \in L^j$ for some $i, j \geqslant 0$. But then $w = u \cdot v \in L^i \cdot L^j = L^{i+j} \subseteq L^*$.

4. Prove that $L^+ \cdot L^+ \subseteq L^+$ for all languages L.

   Solution. Suppose $w \in L^+ \cdot L^+$, that is $w = u \cdot v$ for some $u, v \in L^+$. By the definition of L^+ we have then $u \in L^i$ and $v \in L^j$ for some $i, j \geqslant 1$. But then $w = u \cdot v \in L^i \cdot L^j = L^{i+j} \subseteq L^+$, since $i+j > 0$.

5. For any non-empty language L, $L^+ \subseteq L^+ \cdot L^+$ iff $\varepsilon \in L$.

   Solution. Let L be non-empty.

   If $\varepsilon \in L$ then $\varepsilon \in L^+$, since $L \subseteq L^+$ by the definition of L^+. So $L^+ = \{\varepsilon\} \cdot L^+ \subseteq L^+ \cdot L^+$.

   Conversely, if $\varepsilon \notin L$, let w be a string in L^+ of shortest length. Such w exists, since L is not empty, and is not ε by assuptiojn. The minimal length of strings in $L^+ \cdot L^+$ is thus $2|w|$, which is $> |w|$, because $w \neq \varepsilon$. So $w \in L^+$ but $w \notin L^+ \cdot L^+$.

6. Prove that $(L^+)^+ = L^+$ for all languages L.

   Solution. For any language M we have $M \subseteq M^+$, by the definition of M^+, so $L^+ \subseteq (L^+)^+$.

   To prove that $(L^+)^+ \subseteq L^+$ take $w \in (L^+)^+$. By the definition of +, $w \in (L^+)^n$ for some $n > 0$. Since $L^+ \cdot L^+ \subseteq L^+$ (proved above), we have $(L^+)^n = L^+ \cdot \dots \cdot L^+$ (n times) $\subseteq L^+$. So $w \in L^+$.

   The latter argument can be articulated by referring to strings rather than languages.

   If $w \in (L^+)^+$ then $w = u_1, \dots, u_k \in L^+$ for some $u_1, \dots, u_k \in L^+$. Each u_i is $v_{i1} \cdot \dots \cdot v_{in_i}$ for some $v_{i1} \dots v_{in_i} \in L$. So

   $$w = u_{11} \cdot \dots \cdot u_{1n_1} \cdot u_{21} \cdot \dots \cdot u_{2n_2} \cdot \dots \cdots \cdot u_{k1} \cdot \dots \cdot u_{kn_k}$$

   which is a string in L^+.

7. Give an example of a Σ-language L for which $(L^2)^* \neq (L^*)^2$.