Assignment 12: PTime reductions and NP-completeness

A. For each of the following determine whether it is true, and explain your answer.

(i) If L is NP and $L \leqslant_p L'$ then L' is NP.
Solution. False. Let K be a non-NP language (for example an undecidable language). Take $L' = L \oplus K$. Then $L \leqslant_p L'$, but L' is not NP since $K \leqslant_p L'$.

(ii) If L is NP-hard and $L \leqslant_p L'$ then L' is NP-hard.
Solution. True. If L is NP-hard then (by dfn) every NP-problem is $\leqslant_p L$. By transitivity of $\leqslant_p$, every NP problem is $\leqslant_p L'$.

B. (20%) TWICE-SAT: *Given a boolean expression E, is it satisfied by at least two different valuations.*

Given that BOOL-SAT is NP-hard, prove that TWICE-SAT is NP-complete.

Solution.

- TWICE-SAT is NP: A certificate for an instance E is a pair of different valuations, each satisfying E. The certificate is of size linear in $|E|$, and can be checked in linear time.
- TWICE-SAT is NP-hard: Since BOOL-SAT is NP-hard, it suffices to define a PTime-reduction $\rho :$ BOOL-SAT $\leqslant_p$ TWICE-SAT. Let ρ map an instance E of BOOL-SAT to $E \wedge (y \leftrightarrow z)$ where y, z are fresh variables. ($y \leftrightarrow z$ can be expressed as $(y \wedge z) \vee (-y \wedge -z)$.) ρ is trivially PTime.
  It is a reduction: If E is satisfiable by a valuation V then $\rho(E)$ is satisfied by V extended with $y, z \mapsto 0$ as well as V extended with $y, z \mapsto 1$.
  Conversely, if $\rho(E)$ is satisfied (let alone satisfied twice) then E, being a conjunct of $\rho(E)$, must be satisfied.

C. Consider the decision-problem
3OCCUR-SAT: *Given a boolean expression E with every variable occurring at most three times, is E satisfiable.*

Show that this problem is NP-hard by reducing BOOL-SAT to it.

Solution. Define $\rho :$ BOOL-SAT $\leqslant_p$ 3OCCUR-SAT as follows. Given an expression E with variables $x_1 \ldots x_k$ replace for each $i \in [1..k]$ the occurrences of x_i with fresh distinct variables $x_{i1}, x_{i2} \ldots x_{im_i}$, one for each occurrence of x_i. (m_i is the number of occurrences of x_i). This yields an expression E' where every variable occurs only once.

Now let $\rho(E)$ be E' conjuncted with the expressions asserting that all "copies" x_{ij} of x_i are equivalent, that is all expressions $x_{i1} \to x_{i2}$, $x_{i2} \to x_{i3}$, $\ldots$, $x_{im_i} \to x_{i1}$. By its definition, E' has each variable occurring exactly three times. (If implications are to be avoided, we can just write $-a \vee b$ for $a \to b$.)

ρ is clearly computable in PTime. It is a reduction:
If E has a satisfying valuation V then V' defined by $V'(x_{ij}) = V(x_i)$ satisfies $\rho(E)$.

Conversely, if E' has a satisfying valuation V then V must assign, for each i, the same boolean value to all variables x_{ij}. So E is satisfied by the valuation V that to x_i assigns the value $V(x_{i1}) = \cdots = V(x_{im_i})$.

1. (35%) Given a boolean expression E with an even number of variables, say that a valuation V is ***balanced*** for F if it assigns 1 to half of the variables and 0 to the rest.

BALANCED-SAT: *Given a boolean expression E with an even number of variables is it satisfied by some balanced valuation V.*

Given that **BOOL-SAT** is NP-hard, prove that **BALANCED-SAT** is NP-complete. [Hint: For a single-variable expression $E[x]$ you would conjunct $E[x]$ with $E[-y]$.]

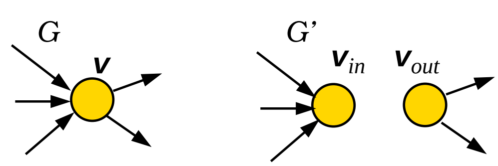
2. (35%) A ***weighted digraph*** is a digraph with each edge assigned a positive integer (its "weight"). The ***weight of a path*** is the sum of its edges' weights.

Consider the problem **WEIGHTED-PATH**:
Given a weighted digraph G and a target integer $t>0$ does G have a path, without repeated vertices, of weight $\geqslant t$.

Define a PTime reduction of **HAMILTONIAN-PATH** to **WEIGHTED-PATH**.

D. Recall the decision-problem
HAMILTONIAN-PATH (HP): *Given a directed graph G, does it have a Hamiltonian-path (H-path), i.e. a path visiting every vertex once.*
The **HAMILTONIAN-CYCLE (HC)** decision-problem asks the same question for a ***cycle***, i.e. a closed loop.

(i) Define a reduction $\rho:$ **HC** $\leqslant_p$ **HP**.
Solution. Given a digraph $G = (V, E)$ Choose any vertex $v \in V$. Let $G' = \rho(G)$ be G with v split into two vertices v_{in} and v_{out}. v_{in} inherits the incoming edges of v, and v_{out} the outgoing edges of v.



ρ is computed in PTime trivially.

Suppose G has a H-cycle. $v \to v_1 \cdots \to v_k \to v$. Then $v_{out} \to v_1 \cdots \to v_k \to v_{in}$ is a H-path in G'.

Conversely, if G' has a H-path then the path's first vertex must be v_{out} (which has no incoming edges) and it must end at v_{in} (which has no outgoing edges). So $v \to v_1 \cdots \to v_k \to v$ is a H-cycle in G .

(ii) Define a reduction $\rho :$ **HP** $\leqslant_p$ **HC**.

Solution. Let ρ map an instance G of **HP** to the di-graph G' obtained by adding to G a new vertex v , and for each vertex u of G an edge from v to u and an edge from u to v.

ρ is clearly computable in PTime. To show that it is a reduction, assume G has a Hamiltonian path $u_1, \cdots, u_k$. Then $v, u_1, \cdots, u_k, v$ is a Hamiltonian cycle in G' .

Conversely, if there is a Hamiltonian cycle in G' , it can be listed starting with v : $v, u_1, \ldots, u_k, v$. Then $u_1, \cdots, u_k$ is a Hamiltonian path in G.

3. (35%) **TWO-CLIQUES**: *Given an undirected graph G and a target $t > 0$, does G have two disjoint cliques of size $\geqslant t$.*

Given that **CLIQUE** is NP-hard, prove that **TWO-CLIQUES** is NP-complete.