Assignment 11: Computable reductions

This assignment contains solved practice problems, numbered in red. The assigned problems and sub-problems are numbered in green.

A. The problem **COMMON-ACCEPT** asks whether a given pair (M_0, M_1) of Turing acceptors accept a common string.

A decidable certification for **COMMON-ACCEPT** has $c \vdash (M_0, M_1)$ iff c is a pair (t_0, t_1) where t_0 and t_1 are accepting traces of M_0 and M_1, respectively, for the same input string. So **COMMON-ACCEPT** is SD.

(i) Define a computable reduction of **ε-ACCEPT** to **COMMON-ACCEPT**.

Solution. Let ρ map an instance M of **ε-ACCEPT** to the instance (E, M) of **COMMON-ACCEPT**, where E is an acceptor for the singleton language $\{\varepsilon\}$. Then M accepts ε iff $\{\varepsilon\} = \mathcal{L}(E) \subseteq \mathcal{L}(M)$, i.e. ρ is a reduction. ρ is computable trivially.

(ii) Conclude that **COMMON-ACCEPT** is not decidable. (This cannot be proved by invoking Rice's Theorem as we stated it, because the instances are here ***pairs*** of acceptors.)

Solution. Since **ε-ACCEPT** is undecidable and computably-reducible to **COMMON-ACCEPT** it follows that the latter is undecidable as well.

1. (60%) The problem **SUBLANG** asks whether a given pair (M, M') of Turing acceptors satisfies $\mathcal{L}(M) \subseteq \mathcal{L}(M')$.

(a) Define a computable reduction of **ε-ACCEPT** to **SUBLANG**.

Solution. Fix an acceptor E for the singleton language $\{\varepsilon\}$. Let ρ be a function that maps an instance M of **ε-ACCEPT** to the instance (E, M) of **SUBLANG**. ρ is clearly computable, as a purely syntactic program modification. M accepts ε iff $\{\varepsilon\} \subseteq \mathcal{L}(M)$, that is iff $\rho(M^\#) = (E, M) \in$ **SUBLANG**, so ρ is a reduction. It is trivially computable.

(b) Define a computable reduction of **ε-NONACCEPT** to **SUBLANG**.

Solution. Given as input an instance $M^\#$ of **ε-NONACCEPT** let $\rho(M^\#)$ be the instance (M, P) of **SUBLANG** where P is an acceptors recognizing Σ^+. Then M fails to accept ε iff $\mathcal{L}(M) \subseteq \Sigma^+$, i.e. iff (M, P) satisfies **SUBLANG**. ρ is trivially computable.

(c) Conclude that neither **SUBLANG** nor its complement are SD. (You may use the fact that **ε-NONACCEPT** is not SD, as proved in class).

Solution. By (a) **SUBLANG** is not SD. And by (i) the complement of **SUBLANG** reduces to **ε-NONACCEPT**, so that complement is not SD either.

2. (40%) Let $\Sigma = \{\mathtt{a}, \mathtt{b}\}$. For Σ-languages L, L' define $L \oplus L' =_{\mathrm{df}} \{\mathtt{a}\} \cdot L \cup \{\mathtt{b}\} \cdot L'$. (The definition given later in class is slightly different. Hopefully you don't find this confusing.)

(a) Define computable reductions $\rho : L \leqslant_c L \oplus L'$ and $\rho' : L' \leqslant_c L \oplus L'$.

Solution. Let $\rho(w) = \mathtt{a} \cdot w$. This is trivially computable, and we have $w \in L$ iff $\mathtt{a} \cdot w \in L \oplus L'$ by the definition of $\oplus$.
Similarly, let $\rho(w) = \mathtt{b} \cdot w$.

(b) Suppose L is SD but not decidable.
Prove that $L \oplus \bar{L}$ is neither SD nor co-SD.

Solution. Since L is SD but not decidable, its complement $\bar{L}$ is not SD, or else L would be decidable. Since $\bar{L} \leqslant_c L \oplus \bar{L}$ it follows that $L \oplus \bar{L}$ is not SD either, or else $\bar{L}$ would be SD.

The complement of $L \oplus \bar{L} = \mathtt{a}L \cup \mathtt{b}\bar{L}$ is $\{\varepsilon\} \cup \mathtt{b}L \cup \mathtt{a}\bar{L}$, to which $\bar{L}$ c-reduce as in (a). So the complement of $L \oplus \bar{L}$ is not SD either, i.e. $L \oplus \bar{L}$ is not co-SD.