B501, Fall 2024

Assignment 10: Undecidability

This assignment contains solved practice problems, numbered in red.
The assigned problems and sub-problems are numbered in green.

1. (15%) Show that the following decision problems are decidable.

(i) Given a Turing acceptor M, does it accept ε within 10^{10} steps?

Solution. Decision algorithm: Run M on input ε for up to 10^{10} steps, and accept M if and when acceptance is reached, reject otherwise.

(a) Given Turing acceptor M, does it accept some string within 10^{10} steps? [Hint: How long are the strings that M can actually read within 10^{10} steps?]

2. (15%) Show that the following decision-problems are undecidable.

(i) Given a Turing acceptor M, does it accept ε?

Solution. This problem asks whether $\varepsilon \in \mathcal{L}(M)$, so it is a scope problem. It is non-trivial because some acceptors accept ε while some do not. So by Rice's Theorem the problem is undecidable.

(a) Given a Turing acceptor M, does it accept some string of length $\leqslant 10^{10}$?

3. (45%) Show that the following decision problems are SD.

(i) Given a Turing acceptor M, does it accept ε?

Solution. Let $\vdash$ be the mapping where $c \vdash M^{\#}$ iff c is an accepting trace of M for input ε. This is clearly a certification, and it is decidable because there is an algorithm to check that c satisfies the conditions stated. Since the problem has a decidable certification, it is SD.

(a) Given a Turing acceptor M, does it accept some string of length $\leqslant 10^{10}$?

(b) Given a Turing acceptor M, does it accept at least two different strings?

(c) Given two Turing-acceptors M_0, M_1 is there is a string accepted by both.

4. (15%) Show that the following decision problems are not SD.

(i) Given a Turing acceptor M, does it accept no string of length $\leqslant 10$?
Solution. Consider the complement problem (as it applies to instances of the problem, i.e. disregarding junk strings):
Given a Turing acceptor M, does it accept some string of length $\leqslant 10$? By Rice's Theorem it is undecidable. But it is SD, since it has a decidable certification, as in problems above. So its complement, i.e. the given problem, cannot be SD, or else that complement problem would be decidable.

(a) Given a Turing acceptor M, does it fail to accept ε?

5. (10%) Prove that if $L \subseteq \Sigma^*$ has a ***semi-decidable*** certification $\vdash_L$, then L is SD. [Hint: Since $\vdash_L$ is SD it has a decidable certification, call it $\vdash_{cert}$. That is, $d \vdash_{cert} (e, w)$ iff $e \vdash_L w$.]