B501, Fall 2024
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Assignment 10: Undecidability

This assignment contains solved practice problems, numbered in red.
The assigned problems and sub-problems are numbered in green.

1. (15%) Show that the following decision problems are decidable.

(i) Given a Turing acceptor M, does it accept ε within 10^{10} steps?

Solution. Decision algorithm: Run M on input ε for up to 10^{10} steps, and accept M if and when acceptance is reached, reject otherwise.

(a) Given Turing acceptor M, does it accept some string within 10^{10} steps? [Hint: How long are the strings that M can actually read within 10^{10} steps?]

Solution. Decision algorithm: Run Turing acceptor M on each string of length $\leqslant 10^{10}$ in turn, each time not exceeding 10^{10} computation steps. Stop and accept if and when an accepting configuration is reached, reject otherwise. This algorithm detects acceptance within 10^{10} steps for arbitrary input strings, because in 10^{10} steps no more than the first 10^{10} symbols of the input can be read.

2. (15%) Show that the following decision-problems are undecidable.

(i) Given a Turing acceptor M, does it accept ε?
Solution. This problem asks whether $\varepsilon \in \mathcal{L}(M)$, so it is a scope problem. It is non-trivial because some acceptors accept ε while some do not. So by Rice's Theorem the problem is undecidable.

(a) Given a Turing acceptor M, does it accept some string of length $\leqslant 10^{10}$?
Solution. This is a non-trivial scope problem, so it is undecidable by Rice's Theorem. Note that the length-bound is immaterial. It was here just to pull your leg...

3. (45%) Show that the following decision problems are SD.

(i) Given a Turing acceptor M, does it accept ε?
Solution. Let $\vdash$ be the mapping where $c \vdash M^\#$ iff c is an accepting trace of M for input ε. This is clearly a certification, and it is decidable because there is an algorithm to check that c satisfies the conditions stated. Since the problem has a decidable certification, it is SD.

(a) Given a Turing acceptor M, does it accept some string of length $\leqslant 10^{10}$?
Solution. This problem has a decidable certification $\vdash$, where $c \vdash M^\#$ iff c is an accepting computation for some string of length $\leqslant 10^{10}$. Note that the certification consisting merely of an accepted string would not do here, because it is not decidable.

(b) Given a Turing acceptor M, does it accept at least two different strings?
Solution. This problem has a decidable certification $\vdash$ where $c \vdash M^\#$ iff c is a pair of accepting traces of M for two different inputs.

(c) Given two Turing-acceptors M_0, M_1 is there is a string accepted by both.
Solution. This problem has a decidable certification $\vdash$ where $c \vdash M^\#$ iff c is a pair consisting of an accepting trace of M_0 for some input w, and an accepting trace of M_1 for the same w as input.

4. (15%) Show that the following decision problems are not SD.

(i) Given a Turing acceptor M, does it accept no string of length $\leqslant 10$?
Solution. Consider the complement problem (as it applies to instances of the problem, i.e. disregarding junk strings):
Given a Turing acceptor M, does it accept some string of length $\leqslant 10$? By Rice's Theorem it is undecidable. But it is SD, since it has a decidable certification, as in problems above. So its complement, i.e. the given problem, cannot be SD, or else that complement problem would be decidable.

(a) Given a Turing acceptor M, does it fail to accept ε?
Solution. This is the complement of the ε-**ACCEPT** problem, which is undecidable, by Rice Theorem, but SD as we showed. Had the given problem been SD then ε-**ACCEPT** would be both SD and co-SD, and therefore decidable. So the given decision-problem is not SD.

5. (10%) Prove that if $L \subseteq \Sigma^*$ has a ***semi-decidable*** certification $\vdash_L$, then L is SD. [Hint: Since $\vdash_L$ is SD it has a decidable certification, call it $\vdash_{cert}$. That is, $d \vdash_{cert} (e, w)$ iff $e \vdash_L w$.]

Solution. Suppose $\vdash_L$ is a SD certification for a language $L \subseteq \Sigma^*$. The relation $\vdash_L$ being SD, there is a decidable certification $\vdash_{cert}$ for it. Consider the decidable binary relation $\vdash$ where $c \vdash w$ iff c is a pair (d, e) with $d \vdash_{cert} (e, w)$. This relation is decidable because $\vdash_{cert}$ is decidable. And it is a certification of L:

$(d, e) \vdash w$

iff	$d \vdash_{cert} (e, w)$	(defn of $\vdash$)
iff	$e \vdash_L w$	(since $\vdash_{cert}$ is a certification for $\vdash_L$)
iff	$w \in L$	(since $\vdash_L$ is a certification for L)