PDAs, Chomsky's Hierarchy

In this assignment, **"construct directly a PDA"** means that no auxiliary stack symbols are used other than the bottom-marker. This implies that you cannot construct your PDA by converting a CFG to it.

Unless otherwise stated, Σ will denote the alphabet $\{\mathtt{a}, \mathtt{b}\}$.

1. (20%) For each of the following statements about languages $L, M, R \subseteq \Sigma^*$ determine whether it is always true, and justify your answer.

(i) If L is CF and R regular, then $L - R$ is CF.
Solution. True. The complement $\bar{R}$ of R must be regular, and so $L - R = L \cap \bar{R}$ must be CF, as the intersection of a CFL and a regular language.

(a) If R is regular and L CF, then $R - L$ is regular.

(b) If R and R' are regular and L is CF then $(R \cdot L) \cup (R' \cdot L)$ is CF.

A. For a string $w \in \{\mathtt{a}, \mathtt{b}, \#\}^*$ let $w^\#$ be the initial substring of w through the first occurrence of $\#$, if there is one. For example, if $w = \mathtt{baa\#aba\#a}$ then $w^\# = \mathtt{baa\#}$ and if $w = \mathtt{abb}$ then $w^\# = \mathtt{abb}$. For a language $L \subseteq \{\mathtt{a}, \mathtt{b}, \#\}^*$ define $L^\# =_{\mathrm{df}} \{w^\# \mid w \in \{\mathtt{a}, \mathtt{b}, \#\}^*\}$.

Show that if L is a CFL then so is $L^\#$.

Solution. Given a CFL L consider a PDA P recognizing L, with f as accepting state. We may assume that f is a terminal state, i.e. there are no transitions of the form $f \xrightarrow{\sigma\,(\alpha\to\beta)} q$. Let $P^\#$ be the PDA P with each transition of the form $q \xrightarrow{\#\,(\alpha\to\beta)} p$ replaced by $q \xrightarrow{\#\,(\alpha\to\beta)} f$. Then $P^\#$ is a PDA that recognizes $L^\#$. So $L^\#$ is a CFL.

B.

(i) Construct directly a PDA that recognizes the language $L = \{\mathtt{a}^i\mathtt{c}(\mathtt{ab})^i \mid i \geqslant 0\}$.

Solution. L is recognized by the following PDA M. M pushes $\mathtt{ab}$'s on the for each consecutive $\mathtt{a}$ it reads. On reading $\mathtt{c}$ M switches to a phase that pops an $\mathtt{a}$ for every $\mathtt{a}$ read and a $\mathtt{b}$ for each $\mathtt{b}$ read.
With states $\{s, q, p_0, p_1, f\}$, s being initial and f accepting, the transition rules are:

$$\begin{array}{lll} s \xrightarrow{\epsilon\,(\epsilon\to\$)} q & \qquad & p_0 \xrightarrow{\mathtt{a}\,(\mathtt{a}\to\epsilon)} p_1 \\ q \xrightarrow{\mathtt{a}\,(\epsilon\to\mathtt{a})} q & & p_1 \xrightarrow{\mathtt{b}\,(\epsilon\to\epsilon)} p_0 \\ q \xrightarrow{\mathtt{c}\,(\epsilon\to\epsilon)} p_0 & & p_0 \xrightarrow{\epsilon\,(\$\to\$)} f \end{array}$$

(ii) The obvious CFG G to generate L is $S \to \mathtt{a}\,S\,\mathtt{ab} \mid \mathtt{c}$. Convert G into another PDA that recognizes L.

Solution. With initial state s and accepting state f:

$$\begin{array}{lllll} s \xrightarrow{\epsilon\,(\epsilon\to S\$)} q & \qquad & q \xrightarrow{\epsilon\,(S\to\mathtt{c})} q & \qquad & q \xrightarrow{\mathtt{b}\,(\mathtt{b}\to\epsilon)} q \\ q \xrightarrow{\epsilon\,(S\to\mathtt{a}S\mathtt{ab})} q & & q \xrightarrow{\mathtt{a}\,(\mathtt{a}\to\epsilon)} q & & q \xrightarrow{\epsilon\,(\$\to\epsilon)} f \end{array}$$

2. (20+20%)

(a) Construct directly a PDA that recognizes the language
$L = \{\mathtt{a}^{p+q}\mathtt{b}^q\mathtt{c}^p \mid p, q \geqslant 0\}$.

(b) Define a CFG that generates L, and then convert it into another PDA N that recognizes L, different from the one in (a).

3. (20%) For this problem read the revised slides for Chomsky's Hierarchy.
Let L be the language $L = \{\, \mathtt{a}^p\mathtt{b}^q\mathtt{c}^p\mathtt{d}^q \mid p, q > 0 \,\}$. Show that L is context-sensitive.
[Hint: Define a non-contracting grammar that generates L, using the same approach as the one used for $\mathtt{a}^n\mathtt{b}^n\mathtt{c}^n$ in the slides.]