B501, Fall 2024

Assignment 7: Context-free languages

Solutions.

Solved practice problems numbered in red, assigned problems in green.

1. (25%) For each grammar below describe in words the language it generates.

(i) $S \to \mathtt{a}S\mathtt{b} \mid \mathtt{b}S\mathtt{a} \mid \varepsilon$

Solution. The set of strings $w\tilde{w}$ where $\tilde{w}$ is w with $\mathtt{a}$ and $\mathtt{b}$ interchanged.

(a) $S \to SS \mid \mathtt{a}$

Solution. $\mathcal{L}(\mathtt{a}^+)$, i.e. the non-empty strings over the letter $\mathtt{a}$.

(ii) $S \to \mathtt{a}S \mid \mathtt{c}T$, $T \to \mathtt{a}T \mid \mathtt{c}R$, $R \to \mathtt{a}R \mid \varepsilon$.

Solution. The strings of the form $\mathtt{a}^i\mathtt{c}\mathtt{a}^j\mathtt{c}\mathtt{a}^k$ where $i,j,k \geqslant 0$.

(b) $S \to \mathtt{a}A \mid \mathtt{b}B$, $A \to \mathtt{a}A \mid \mathtt{b}A \mid \mathtt{a}$, $B \to \mathtt{a}B \mid \mathtt{b}B \mid \mathtt{b}$.

Solution. The strings that start and end with the same letter, i.e. the strings of the form $\mathtt{a} \cdot w \cdot \mathtt{a}$ and the strings of the form $\mathtt{b} \cdot w \cdot \mathtt{b}$.

2. (75%) For each of the following languages give a CFG that generates it.

(a) $L = \{\mathtt{a}^n\mathtt{b}^p\mathtt{c}^q \mid n = p+q\}$

Solution. $S \to \mathtt{a}S\mathtt{c} \mid B$, $B \to \mathtt{a}B\mathtt{b} \mid \varepsilon$

(i) $L = \{\mathtt{a}^n x \mid x \in \{\mathtt{a},\mathtt{b}\}^*, |x| = n\}$

Solution. $S \to \mathtt{a}S\mathtt{a} \mid \mathtt{a}S\mathtt{b} \mid \varepsilon$

(b) $L = \{\mathtt{a}^n x \mid \#_a(x) = n\}$

Solution. $S \to \mathtt{a}S\mathtt{a} \mid S\mathtt{b} \mid \varepsilon$.

(ii) $L = \{\mathtt{a}^i\mathtt{b}^i\mathtt{c}^j\mathtt{d}^j \mid i,j \geqslant 0\}$

Solution. $S \to LR$, $L \to \mathtt{a}L\mathtt{b} \mid \varepsilon$, $R \to \mathtt{c}R\mathtt{d} \mid \varepsilon$

(c) $L = \{\mathtt{a}^i\mathtt{b}^j\mathtt{c}^j\mathtt{d}^i \mid i,j \geqslant 0\}$

Solution. $S \to \mathtt{a}S\mathtt{d} \mid M$, $M \to \mathtt{b}M\mathtt{c} \mid \varepsilon$

(d) $L = \{\mathtt{a}^p\mathtt{b}^{p+2q} \mid p,q \geqslant 0\}$

Solution. $S \to \mathtt{a}S\mathtt{b} \mid S\mathtt{b}\mathtt{b} \mid \varepsilon$.

(e) $L = \{\mathtt{a}^n\mathtt{b}^m\mathtt{c}^p\mathtt{d}^q \mid m+n = p+q, n \geqslant q\}$.
Hint: $L = \{\mathtt{a}^{q+i}\mathtt{b}^m\mathtt{c}^{m+i}\mathtt{d}^q \mid m, q, i \geqslant 0\}$
Solution. $S \to \mathtt{a}S\mathtt{d} \mid T, \quad T \to \mathtt{a}T\mathtt{c} \mid U, \quad U \to \mathtt{b}U\mathtt{c} \mid \varepsilon$.

(f) $X = \{\mathtt{a}^n\mathtt{b}^m\mathtt{c}^p\mathtt{d}^q \mid m+n = p+q\}$.
Hint: This is the union of L above and a similar language with $n \leqslant q$.
Solution. $X = L \cup L'$ where L is as in the previous problem (e) and

$$\begin{aligned} L' &= \{\mathtt{a}^n\mathtt{b}^m\mathtt{c}^p\mathtt{d}^q \mid m+n = p+q, q \geqslant n\} \\ &= \{\mathtt{a}^q\mathtt{b}^{m+i}\mathtt{c}^m\mathtt{d}^{q+i} \mid m+n = p+q\} \end{aligned}$$

So X is generated by the CFG

$$\begin{aligned} &S \to N \mid Q \\ &\quad N \to \mathtt{a}N\mathtt{d} \mid T, \quad T \to \mathtt{a}T\mathtt{c} \mid U, \quad U \to \mathtt{b}U\mathtt{c} \mid \varepsilon \\ &\quad Q \to \mathtt{a}Q\mathtt{d} \mid P, \quad P \to \mathtt{b}P\mathtt{d} \mid V, \quad V \to \mathtt{b}V\mathtt{c} \mid \varepsilon \end{aligned}$$