Assignment 5: Regular languages

Solutions.

1. (14%) For each of the following languages over $\Sigma = \{\mathtt{a}, \mathtt{b}\}$ show that it is basic, by giving regular expressions and/or using closure properties of the basic languages. Do not use recognition by DFAs or NFAs. Assume that $\star$ is an extra regular expression denoting Σ^*.

(i) The set of strings with no consecutive a's.

Solution. The set of strings with two consecutive a's is denoted by the regular expression $\star\mathtt{aa}\star$ and is therefore basic. The given language is its complement, and is therefore basic as well.

(a) $L = \{w \in \Sigma^* \mid \#_a(w) \text{ even and } \#_b(w) \text{ odd} \}$

Solution. $\{w \in \Sigma^* \mid \#_a(w) \text{ even}\}$ is recognized by a two state automaton, as we saw, Similarly, $\{w \in \Sigma^* \mid \#_b(w) \text{ even}\}$ is regular, and therefore its complement is regular. $\{w \in \Sigma^* \mid \#_b(w) \text{ odd}\}$ is regular. Since the former and latter languages are both regular, so is the given one, which is their intersection.

(b) The language L consisting of strings with no substring $\mathtt{aaaa}$ or $\mathtt{bbbb}$.

Solution. The language with 4 consecutive a's or 4 consecutive b's is denoted by the regular expression $(\star \cdot \mathtt{aaaa} \cdot \star) \cup (\star \cdot \mathtt{bbbb} \cdot \star)$, and is therefore regular. The given language is its complement and is therefore regular as well.

2. (16%) Suppose that $L \subseteq \Sigma^*$ is basic. Use closure under set and language operations to show that the following languages are also basic.

(a) $L' = \{w \in L \mid |w| \text{ is even } \}$

Solution. $L' = L \cap (\Sigma \cdot \Sigma)^*$. L and Σ are regular, and the collection of regular languages is closed under concatenation, star, and intersection. So L' is regular.

(b) $\tilde{L} = \{x_1 \cdot y_1 \cdots \cdots x_n \cdot y_n \mid n \geqslant 0, x_i \in L,\ y_i \notin L\}$

Solution. $\tilde{L} = (L \cdot (\Sigma^* - L))^*$. L and Σ are regular, and the collection of regular languages is closed under star, difference, and concatenation. So $\tilde{L}$ is regular.

3. (10%)

(a) Given a language K describe an infinite collection $L_1, L_2, \ldots$ of basic languages whose union is K.

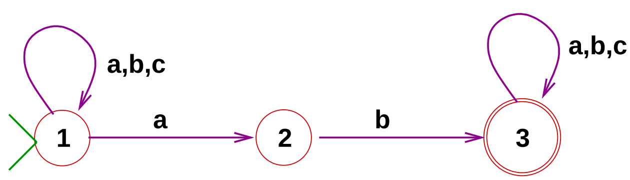
Solution. Let $L_1, L_2, \ldots$ be a listing of the singleton languages $\{w\}$ for $w \in K$. Singletons are finite, and therefore basic, and the union of this listing is K.

(b) Given a language K describe an infinite collection of basic languages whose ***intersection*** (i.e. the strings that are in all of them) is K.
[Hint: This problem is dual to the previous one. But in place of the union of trivial finite languages, consider here the intersection of trivial co-finite languages.]

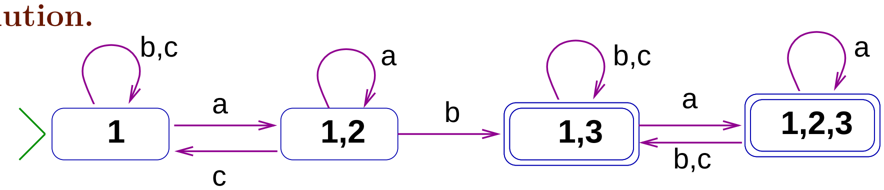
Solution. Let $L_1, L_2, \ldots$ be a listing of the languages $\Sigma^* - \{w\}$ for $w \notin K$. These are all co-finite languages, and therefore basic, but their intersection is K.

4. (10%) Let $\Sigma = \{\texttt{a}, \texttt{b}, \texttt{c}\}$ and $f : \Sigma^* \to \Sigma^*$ the function that for input w yields the string obtained by duplicating each $\texttt{a}$. E.g. $f(\texttt{baaca}) = \texttt{baaaacaa}$. Prove that if L is a regular language, then so is $\{f(w) \mid w \in L\}$. [Hint: Think of a regular expression for L.]

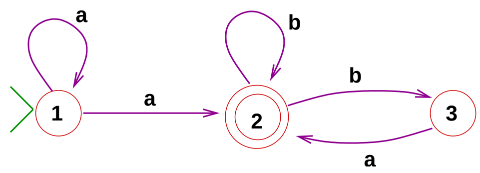
5. (20%) Convert the following NFA into an equivalent DFA.



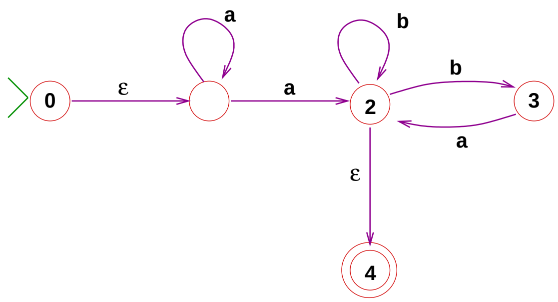
Solution.

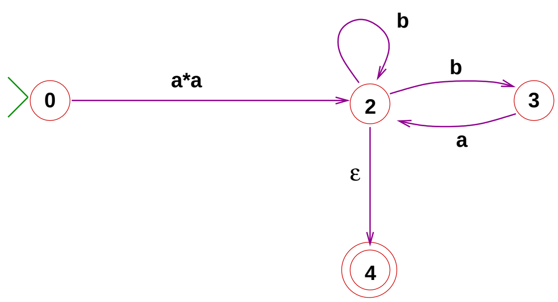


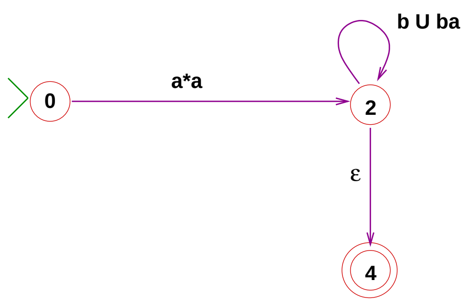
6. (20%) Convert the following NFA into an equivalent regular expression. Exhibit all stages of the conversion.

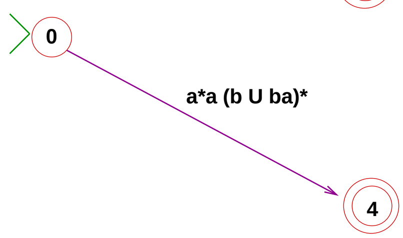


Solution.









7. (10%) A ***CNFA*** (conjunctive NFA) C (over alphabet Σ) is like an NFA, except that a string w is accepted by C if *every* state p such that $s \xrightarrow{w} p$ is accepting. Prove that a language is recognized by a CNFA iff it is regular. [Hint: When is a string w *not* accepted by C?]

Solution. Given an NFA $N = (\Sigma, Q, s, A, \Delta)$ define its *dual* to be the CNFA $\widehat{N} = (\Sigma, Q, s, Q-A, \Delta)$. N accepts a string w iff there is some $a \in A$ such $s \xrightarrow{w} a$, which happens iff it is not the case that $s \xrightarrow{w} b$ for all $b \in Q-A$, i.e. exactly when $\widehat{N}$, as a CNFA, does not accept w. Thus $\mathcal{L}(N) = \Sigma^* - \mathcal{L}(\widehat{N})$.

Therefore $L \subseteq \Sigma^*$ is regular iff $\bar{L} \equiv \Sigma^* - L$ is regular, i.e. iff $\bar{L}$ is recognized by some NFA N, which by the observation above is equivalent to L being recognized by the CNFA $\widehat{N}$.