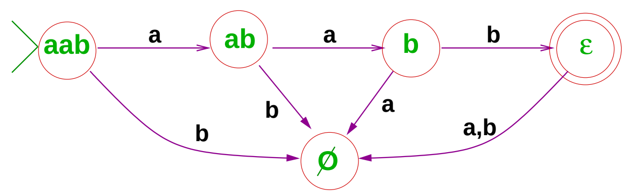
Assignment 4: Automata and residues

1. (20%) Let $L \subseteq \{\mathtt{a}, \mathtt{b}\}^*$ consist of the strings with a in ***some*** odd position. Identify the residues of L and build a DFA from them.

2. (30%) For each of the following languages build an automaton that recognizes it.

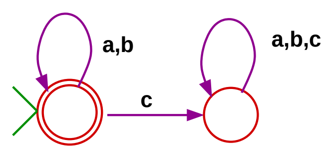
(i) $\{\mathtt{aab}\}$.

Solution.



(ii) $\{\mathtt{a}, \mathtt{b}\}^*$, $\Sigma = \{\mathtt{a}, \mathtt{b}, \mathtt{c}\}$.

Solution.



(a) $\{\mathtt{aba}^n \mid n \geqslant 0\}$.

(b) $\{w \in \{\mathtt{a}, \mathtt{b} \mid$ every a in w is followed by a b $\}$.

3. (20%) We constructed in class an automaton M that recognizes

$\{\{w \in \{\mathtt{a}, \mathtt{b} \mid \#_a(w)$ is even $\}$.

Using the product of your automaton from (1.c) with M construct an automaton that recognizes the intersection of the two languages, i.e. M should accept a string iff it has an even number of a's ***and*** in which every a is followed by a b.

4. (30%) For each of the following languages prove that it is not recognized by any automaton.

(a) $\{a^p b^q \mid 0 \leqslant p \leqslant q\}$.

(b) $\{a^p b^q \mid p \neq q\}$.

(i) $\{x \cdot x^R \mid x \in \{a, b\}^*\}$ (x^R is the reverse of x.)

Solution. We show that L fails the Clipping Property.
Let $k > 0$. Take $w = a^k bba^k$ and u the initial substring a^k of w.
We have $w \in L$ and $|u| \geqslant k$.

If $y = a^p$ is any non-empty substring of u,
the string w' obtained from w by clipping y is of the form $a^{k-p}bba^k$, Such a string cannot be a palindrome, becauise its first half has two b's and its second half has none. So L fails the Clipping Property, and is not recognized by any automaton.

(c) $\{w \in \{a, b, c\}^* \mid \#_a(w) + \#_b(w) = \#_c(w)\}$.

Remark. The arguments above uses directly the failure of the Clipping Theorem. It is common to give such arguments referring to an assumed automaton. In that alternative style, the latter proof might be rendered as follows. This alternative is acceptable, if you prefer it.

Towards contradiction suppose that L is recognized by an automaton M, and let k be its number of states. Let $w = a^k c^k = a^k b^0 c^{k+0}$, which is L. Take $u = a^k$, the initial substring of w.

By the Clipping Theorem u has a non-empty substring $y = a^\ell$ $(\ell > 0)$ so that the string w' obtained from w by removing y is accepted by M. But w' is $a^{k-\ell}c^k$, which is not in L since $\ell > 0$, and therefore not accepted by M.

Since this is a contradiction, the assumption that an automaton M recognizing L exists is false.