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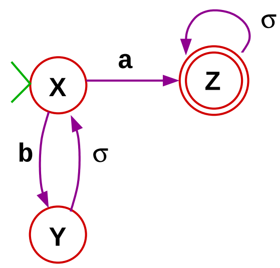
Assignment 4: Automata and residues

Solutions.

1. (20%) Let $L \subseteq \{\mathtt{a}, \mathtt{b}\}^*$ consist of the strings with $\mathtt{a}$ in ***some*** odd position. Identify the residues of L and build a DFA from them.

Solution.



X: a in some odd position

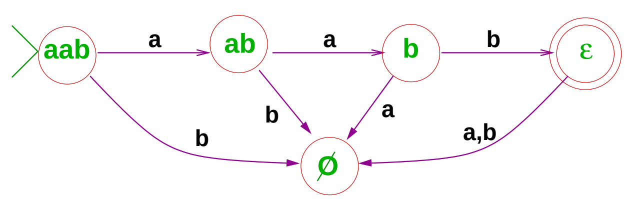
Y: a in some even position

Z: Σ^*

2. (30%) For each of the following languages build an automaton that recognizes it.

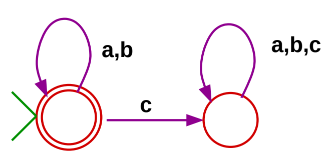
(i) $\{\mathtt{aab}\}$.

Solution.



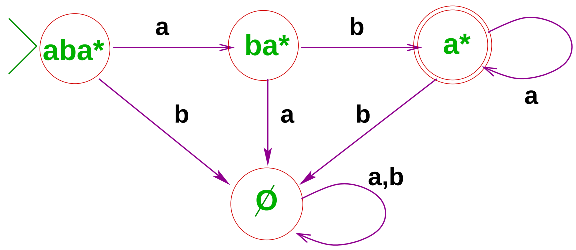
(ii) $\{\mathtt{a}, \mathtt{b}\}^*$, $\Sigma = \{\mathtt{a}, \mathtt{b}, \mathtt{c}\}$.

Solution.



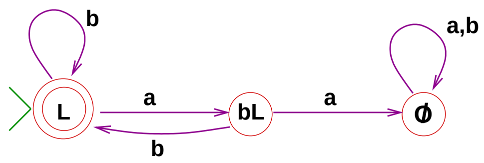
(a) $\{\texttt{aba}^n \mid n \geqslant 0\}$.

Solution.



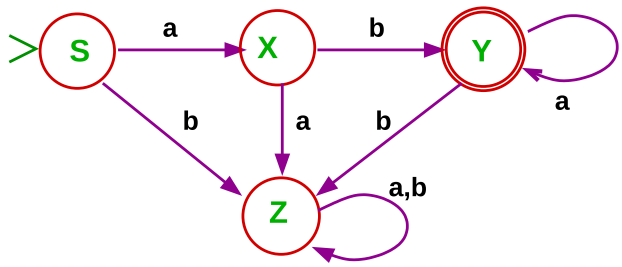
(b) The strings in $\{\texttt{a}, \texttt{b}\}^*$ where every $\texttt{a}$ is followed by a $\texttt{b}$.

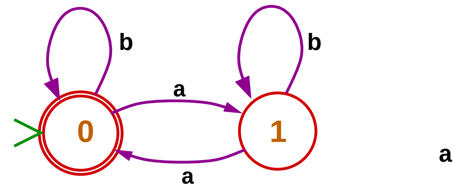
Solution.

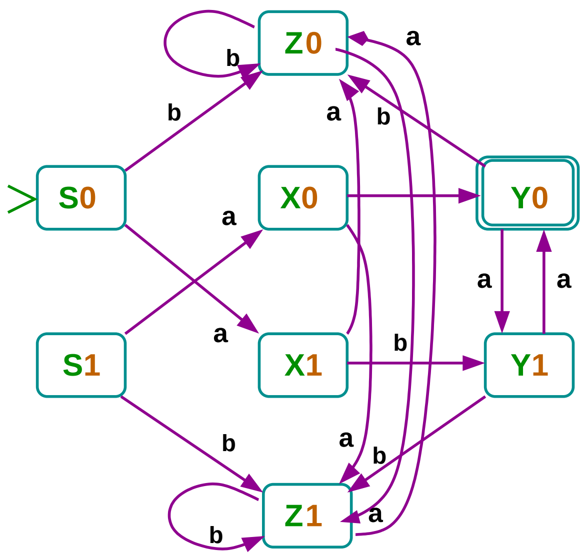


3. (20%) We constructed in class an automaton M that recognizes $\mathcal{L}(\mathtt{a} \cdot \mathtt{b} \cdot \mathtt{a}^{\star})$. Using the product of your automaton above construct an automaton that recognizes the intersection of the two languages, i.e. accepting a string iff it has an even number of `a`'s ***and*** has form $\mathtt{aba}^n$ for some n. [The posted question refered to the language where every `a` is followed by a `b`, but the solution here refers to the problem above, as was intended. Aplogies!].

Solution.







4. (30%) For each of the following languages prove that it is not recognized by any automaton.

(a) $\{\mathtt{a}^p\mathtt{b}^q \mid 0 \leqslant p \leqslant q\}$.

Solution. We show that the language fails the Clipping Property. Let $k > 0$. Consider $w = \mathtt{a}^k\mathtt{b}^{3k}$ and $u = \mathtt{a}^k$ the initial substring of w.
For any non-empty substring $y = \mathtt{a}^\ell$ of u the clipping reduct w' obtained from w by removing y is of the form $\mathtt{a}^\ell\mathtt{b}^{3k}$ with $\ell < k$, which is not in L.

(b) $L = \{\mathtt{a}^p\mathtt{b}^q \mid p \neq q\}$.

Solution. Suppose L were recognized. Then its complement $\bar{L}$ would be recognized, and $\{\mathtt{a}^n\mathtt{b}^n \mid n \geqslant 0\}$ would be recognized, being the intersection of the recognized languages $\bar{L}$ and $\{\mathtt{a}^i\mathtt{b}^j \mid i,j \geqslant 0\}$. But we know that $\{\mathtt{a}^n\mathtt{b}^q \mid p = q\}$ is not recognized, contradiction.

(i) $\{x \cdot x^R \mid x \in \{\mathtt{a},\mathtt{b}\}^*\}$ (x^R is the reverse of x.)

Solution. We show that L fails the Clipping Property.
Let $k > 0$. Take $w = \mathtt{a}^k\mathtt{bba}^k$ and u the initial substring $\mathtt{a}^k$ of w.
We have $w \in L$ and $|u| \geqslant k$.
If $y = \mathtt{a}^p$ is any non-empty substring of u,
the string w' obtained from w by clipping y is of the form $\mathtt{a}^{k-p}\mathtt{bba}^k$, Such a string cannot be a palindrome, because its first half has two $\mathtt{b}$'s and its second half has none. So L fails the Clipping Property, and is not recognized by any automaton.

(c) $\{w \in \{\mathtt{a},\mathtt{b},\mathtt{c}\}^* \mid \#_a(w) + \#_b(w) = \#_c(w)\}$.

Solution. We show that L fails the Clipping Property.
Let $w = \mathtt{a}^k\mathtt{c}^k$ (no $\mathtt{b}$'s), and u the substring $\mathtt{a}^k$ of w. We have $w \in L$ and $|u| \geqslant k$.
If $y = \mathtt{a}^p$ is a non-empty substring of u then the string w' obtained from w by clipping y is $\mathtt{a}^{k-p}\mathtt{c}^k$, which is not in L. So L fails the Clipping Property, and is not recognized by any automaton.

Remark. The arguments above uses directly the failure of the Clipping Theorem. It is common to give such arguments referring to an assumed automaton. In that alternative style, the latter proof might be rendered as follows. This alternative is acceptable, if you prefer it.

Towards contradiction suppose that L is recognized by an automaton M, and let k be its number of states. Let $w = \mathtt{a}^k\mathtt{c}^k = \mathtt{a}^k\mathtt{b}^0\mathtt{c}^{k+0}$, which is L. Take $u = \mathtt{a}^k$, the initial substring of w.
By the Clipping Theorem u has a non-empty substring $y = \mathtt{a}^\ell$ ($\ell > 0$) so that the string w' obtained from w by removing y is accepted by M. But w' is $\mathtt{a}^{k-\ell}\mathtt{c}^k$, which is not in L since $\ell > 0$, and therefore not accepted by M.
Since this is a contradiction, the assumption that an automaton M recognizing L exists is false.