# Assignment 3: Generated sets

B501, Fall 2024


(Due by EOD F Sep 27)

**1.** (15%) Prove by Shifted Induction that for every natural number $n \geqslant 8$ there are $a, b \in \mathbb{N}$ such that $n = 3a + 5b$.

**2.** (10%)

**(i)** Prove by Shifted Induction that $2^n > 2n+1$ for $n \geqslant 3$. (Remember that $2^{k+1} = 2^k + 2^k$.)

**Solution.** Base. For $n = 3$ we have $2^n = 8 > 7 = 2n + 1$.

Step.

Suppose $2^k \geqslant 2k+1$. Then, for $n = k+1$ we have

$$\begin{array}{rcll} 2^n = 2^{k+1} & = & 2^k + 2^k & \\ & > & 2(2k+1) & \text{(IH)} \\ & > & 2(k+1)+1 & \text{since } k > 1 \\ & = & 2n+1 & \end{array}$$

**(a)** Use (i) to prove by Shifted Induction that $2^n > n^2$ for $n \geqslant 5$.

**3.** (15%) Show that for every list $\ell = (a_1 \dots a_n)$ ($n \geqslant 0$) of positive real numbers, if $\prod \ell = a_1 \times \cdots \times a_n = 1$ then $\sum \ell = a_1 + \cdots + a_n \geqslant n$ .
[Hint: For the induction step, when proving for $\ell$ of length $k+1$, let $a$ be the largest entry in $\ell$ and $b$ the smallest, replace $a$ and $b$ by their product $ab$; observe that $a \geqslant 1 \geqslant b$.]

**4.** (15%) Consider checker-boards. Define an L-***piece*** to be three squares forming the shape L. Prove that every $2^n \times 2^n$ board with one square removed can be covered by L-pieces. For example, a $2 \times 2$ board with one squre removed is already a single L-piece!

**5.** (10%) Prove by shifted induction from 1: *If* $A_1, \dots, A_n$ *are sets, of which every two are comparable, then there is an* $A_i$ *which is a subset of all the others.*
(We say that sets $A, B$ are ***comparable*** if either $A \subseteq B$ or $B \subseteq A$.)

**6.** (10%) Use induction *for binary trees* (NOT induction for natural numbers!) to prove that the number of leaves in a binary tree is 1 + the number of internal nodes.

**7.** (10%) Define by recurrence on $\mathbb{N}$ the function $F : \mathbb{N} \rightarrow \text{ASCII}^*$ given by $F(n) = \mathtt{0}^n\mathtt{abc1}^{2n}$. You may use without definition the concatenation function between strings.

**8.** (15%) Fix an alphabet $\Sigma$. The concatenation function over $\Sigma^*$ is defined by the recurrence $\varepsilon \cdot v = v$ , $\sigma u \cdot v = \sigma(u \cdot v)$.
In class we gave a definition by recurrence of the length function $w \mapsto |w|$ from $\Sigma^*$ to $\mathbb{N}$.

Use these definitions to prove by induction on strings that $|w \cdot v| = |w| + |v|$ for all $w, v \in \Sigma^*$.