Assignment 3: Generated sets

B501, Fall 2024

(Due by EOD F Sep 27)

Solutions.

1. (15%) Prove by Shifted Induction that for every natural number $n \geqslant 8$ there are $a, b \in \mathbb{N}$ such that $n = 3a + 5b$.

Solution. Base. For $n = 8$ we can take $a = b = 1$.

Step. Suppose the given property holds for $n = k \geqslant 8$, that is $k = 3a + 5b$ for some $a, b \in \mathbb{N}$. If $b \geqslant 1$ then $k+1 = 3(a+2) + 5(b-1)$. Otherwise, i.e. $b = 0$, we have $k = 3a \geqslant 8$, so $a \geqslant 3$. We have then $k + 1 = 3(a - 3) + 5 \cdot 2$.

By shifted induction it follows that for every natural number $n \geqslant 8$ there are $a, b \in \mathbb{N}$ such that $n = 3a + 5b$.

2. (10%)

(i) Prove by Shifted Induction that $2^n > 2n + 1$ for $n \geqslant 3$. (Remember that $2^{k+1} = 2^k + 2^k$.)

Solution. Base. For $n = 3$ we have $2^n = 8 > 7 = 2n + 1$.

Step.

Suppose $2^k \geqslant 2k+1$. Then, for $n = k + 1$ we have

$$\begin{aligned} 2^n = 2^{k+1} &= 2^k + 2^k \\ &> 2(2k+1) \quad (IH) \\ &> 2(k+1)+1 \quad \text{since } k > 1 \\ &= 2n + 1 \end{aligned}$$

(a) Use (i) to prove by Shifted Induction that $2^n > n^2$ for $n \geqslant 5$.

Solution. ***Base.*** For $n = 5$ we have $2^n = 32 > 25 = n^2$.

Step. Suppose $2^k > k^2$. Then, for $n = k+1$, $k \geqslant 5$,

$$\begin{aligned} 2^n = 2^{k+1} &= 2^k + 2^k \\ &> k^2 + 2^k \quad (IH) \\ &> k^2+2k+1 \quad (by previous observation) \\ &= (k+1)^2 = n^2 \end{aligned}$$

By shifted induction, it follows that $2^n > n^2$ for all natural numbers $n \geqslant 5$.

3. (15%) Show that for every set $R = \{a_1 \ldots a_n\}$ $(n \geqslant 1)$ of positive real numbers, if $\prod R = a_1 \times \cdots \times a_n = 1$ then $\sum R = a_1 + \cdots + a_n \geqslant n$.
[Hint: For the induction step, when proving for R of length $k+1$, let a be the largest entry in R and b the smallest, replace a and b by their product ab; observe that $a \geqslant 1 \geqslant b$.]

Solution. We use Shifted Induction from 1.

Base. $n=1$, say $R=\{a\}$. Then $\prod R=1$ implies $\sum R=a=1\geqslant n$.

▶ Step. Assume our claim holds for sets with $k\geqslant 1$ elements. Given a list R of $k+1$ positive reals such that $\prod R=1$. let a be R's largest entry and b its smallest entry. So $b\leqslant 1$. a and b must be different, because $k+1\geqslant 2$. So $a>1>b$ and therefore $a>ab>b$.

Consider the set $R'=_{\text{df}} R$ with a,b removed and replaced by $a\cdot b$. a and b are different, so R' has k elements. Also, $\prod R'=\prod R=1$. So by IH $\sum R'\geqslant k$. We obtain

$$\begin{aligned}\textstyle\sum R' &= (\textstyle\sum R)+a+b-ab \\ &\geqslant (\textstyle\sum R)+1 && \text{because } a-ab=a(1-b)\geqslant 1-b, \\ & && \text{since } a\geqslant 1 \text{ and } 1-b\geqslant 0 \\ &\geqslant k+1\end{aligned}$$

This concludes the Induction Step and the proof.

Remark. A remarkable consequence of the statement above is that the geometric mean of a set of positive real numbers is $\leqslant$ its arithmetic mean. Given a set R of n positive real numbers, its geometric mean is $G=(\prod R)^{1/n}$. Let $R'=_{\text{df}}\{x/G\mid x\in R\}$. Then

$$\textstyle\prod R'=(\prod R)/G^n=(\prod R)/(\prod R)=1$$

By the statement above, we conclude $\sum R'\geqslant n$.
But $\sum R'=(\sum R)/G$, and so $(\sum R)/G\geqslant n$, that is $(\sum R)/n\geqslant G$.

4. (15%) Consider square checker-boards of unspecified size. Define an L-***piece*** to be three squares forming the shape L. Prove that every $2^n\times 2^n$ board with one square removed can be covered by L-pieces. For example, a 2×2 board with one squre removed is already a single L-piece!

SOLProof by induction on n.

Basis $n=0$, so $2^n=1$. Removing a square from a 1×1 board yields a vacuous board, which is indeed covered by 0 many L-pieces.

Step. Assume the statement true for n. A $2^{n+1}\times 2^{n+1}$ board B can be partitioned into four $2^n\times 2^n$ boards. Removing a square from B results in one of the four quarters having that square removed, while an L-piece in the center of B overlaps each of the three remaining quarters. Removing that piece we get each of the four quarters missing a square, and therefore covered by L-pieces by IH. Together with the L-piece in the middle we obtain a cover of B (with one square removed) by L-pieces.

5. (10%) Prove by shifted induction from 1: *If* $A_1, \ldots, A_n$ *are sets, of which every two are comparable, then there is an* A_i *which is a subset of all the others.*
(We say that sets A, B are ***comparable*** if either $A \subseteq B$ or $B \subseteq A$.)

Solution. Shifted induction on n, starting with 1.

Base: If $n = 1$ then A_1 is a subset of itself, and therefore of all the sets $A_1, \ldots, A_n$.

Step: Suppose the statement holds for $n = k$. Let $A_1, \ldots, A_{k+1}$ be pairwise comparable sets. Consider the list $A_1, \ldots, A_k$. By IH there is an A_i $(i \leqslant k)$ contained in all of them.

By assumption A_i and A_{k+1} are comparable. If $A_i \subseteq A_{k+1}$ then A_i is contained in each of $A_1, \ldots, A_{k+1}$. Otherwise $A_{k+1} \subseteq A_i$, and since the subset relation is transitive, we have $A_{k+1} \subseteq A_j$ for all A_j's. In either case, one of the sets listed is contained in all the others.

By Induction on $\mathbb{N}$ it follows that the statement is true for all $n \in \mathbb{N}$.

6. (10%) Use induction *for binary trees* (NOT induction for natural numbers!) to prove that the number of leaves in a binary tree is 1 + the number of internal nodes.

Solution. Write $L(t)$ for the number of leaves of a binary tree t,
(t) for the number of internal nodes,
and $\langle t_0, t_1 \rangle$ for the tree with t_0 and t_1 as immediate subtrees.

Base: $t = \varepsilon$. There is one leaf (the only node), and no internal node. So $L(t) = 1 = 0 + 1 = I(t) + 1$

Joining trees: Assume the statement true for $t = u_0$ and $t = u_1$. We prove it for $t = \langle u_0, u_1 \rangle$.

$$\begin{aligned} L(u) &= L(u_0) + L(u_1) \\ &= (I(u_0) + 1) + (I(u_1) + 1) \quad \text{(IH)} \\ &= (I(u_0) + I(u_1) + 1) + 1 \\ &= I(u) + 1 \end{aligned}$$

7. (10%) Define by recurrence on $\mathbb{N}$ the function $F : \mathbb{N} \to \textsc{ascii}^*$ given by $F(n) = \mathtt{0}^n\mathtt{abc}\mathtt{1}^{2n}$. You may use without definition the concatenation function between strings.

Solution.
$$\begin{aligned} F(0) &= \mathtt{abc} \\ F(\mathtt{s}x) &= \mathtt{0} \cdot F(x) \cdot \mathtt{11} \end{aligned}$$

8. (15%) Fix an alphabet Σ. The concatenation function over Σ^* is defined by the recurrence $\varepsilon \cdot v = v$, $\sigma u \cdot v = \sigma(u \cdot v)$.

In class we gave a definition by recurrence of the length function $w \mapsto |w|$ from Σ^* to $\mathbb{N}$.

Use these definitions to prove by induction on strings that $|w \cdot v| = |w| + |v|$ for all $w, v \in \Sigma^*$.

Solution. We prove by induction on w that for all v, $|w \cdot v| = |w| + |v|$.

Basis: $w = \varepsilon$:

$$
\begin{aligned}
|\varepsilon \cdot v| &= |v| && \text{(Dfn by recurrence of } \cdot\text{)} \\
&= 0 + |v| \\
&= |\varepsilon| + |v| && \text{(Dfn by recurrence of } |\cdot|\text{)}
\end{aligned}
$$

Steps: $w = \sigma x$:

Assume that for all v, $|x \cdot v| = |x| + |v|$ (IH). Then

$$
\begin{aligned}
|(\sigma x) \cdot v| &= |\sigma(x \cdot v)| && \text{(Dfn of } \cdot\text{)} \\
&= 1 + |x \cdot v| && \text{(Dfn of } |\cdot|\text{)} \\
&= 1 + (|x| + |v|) && \text{(IH)} \\
&= (1 + |x|) + |v| \\
&= |\sigma x| + |v| && \text{(Dfn of } |\cdot|\text{)}
\end{aligned}
$$