Fall 2024

Assignment 2: Relations and Mappings

This assignment contains solved practice problems, numbered in red.
The assigned problems and sub-problems are numbered in green.

1. (25%) Let $A = \{\mathtt{a}, \mathtt{b}, \mathtt{c}, \mathtt{d}\}$ and $B = \{\mathtt{0}, \mathtt{1}, \mathtt{2}\}$. For each of the following types of mapping from A to B determine the number of possible distinct mappings of that type.

(i) All mappings.
Solution. There are 12 elements (pairs) in $A \times B$, so there are $2^{12} = 4048$ possible mappings, i.e binary relations.
Alternative approach: For each $x \in A$ there are $2^3 = 8$ options for output-set. So altogether we have $8^4 = 4048$ mappings.

(ii) Partial functions, i.e. univalent mappings.
Solution. For each $x \in A$ there are four options for $F(x)$: $\mathtt{0}, \mathtt{1}, \mathtt{2}$ and *undefined*. So there are $4 \times 4 = 16$ partial-functions from A to B.

(a) Total-functions.

(b) Total mappings. [Hint: Consider the second solution to (i), but now $\emptyset$ is not an an acceptable output-set.]

(c) Surjective mappings. [Hint: Use (b)]

(d) Injective mappings. [Hint: Same as the number of univalent mappings from B to A. Now (ii).]

(e) Bijections. [Hint: This is a trick question.]

2. (10+10+5%) Let $f : \mathbb{N} \to A$ be an injection, and B a set.

(a) Define an injection $g : \mathbb{N} \times B \to A \times B$.

(b) Define an injection $j : \mathcal{P}(\mathbb{N}) \to \mathcal{P}(A)$.

(c) Define a surjective partial-function $g : A \rightharpoonup \mathbb{N}$.

3. (10+5%) Functions f, g over $\mathbb{N}$ are *almost equal* (notation: $f =_{ae} g$) if there are only finitely many n's for which $f(n) \neq g(n)$.

(a) Prove that $=_{ae}$ is an equivalence relation.

(b) What is the equivalence class of the constant function $f(x) = 0$?

4. (10%) Prove that $\mathbb{R} \times \mathbb{R} \preccurlyeq \mathbb{R}$. [Hint: Define an injection $j : (0..1) \times (0..1) \to (0..1)$ where $j(a, b)$ has the infinite binary expansion obtained by merging the binary expansions of a and of b.]

5. (10%) Show that the set F of functions from $\mathbb{N}$ to $\mathbb{N}$ is not countable. [Hint: Show that $(0..1) \preccurlyeq F$ by mapping each a in the domain to function from $\mathbb{N}$ to $\{0, 1\}$ corresponding to the infinite binary expansion of a.]

6. (15%) Use the CBS Theorem to show that $\{\mathsf{a}, \mathsf{b}\}^* \cong \{\mathsf{a}, \mathsf{b}, \mathsf{c}\}^*$.
[Hint: For an injection $h : \{\mathsf{a}, \mathsf{b}, \mathsf{c}\}^* \to \{\mathsf{a}, \mathsf{b}\}^*$ use two-letter codes for a,b,c. (This is analogous to the binary coding of ASCII characters.)]