Assignment 2: Relations and Mappings (Solutions)

This assignment contains solved practice problems, numbered in red.
The assigned problems and sub-problems are numbered in green.

1. (25%) Let $A = \{\mathtt{a}, \mathtt{b}, \mathtt{c}, \mathtt{d}\}$ and $B = \{\mathtt{0}, \mathtt{1}, \mathtt{2}\}$. For each of the following types of mapping from A to B determine the number of possible distinct mappings of that type.

(i) All mappings.

Solution. There are 12 elements (pairs) in $A \times B$, so there are $2^{12} = 4048$ possible mappings, i.e binary relations.

Alternative approach: For each $x \in A$ there are $2^3 = 8$ options for output-set. So altogether we have $8^4 = 4048$ mappings.

(ii) Partial functions, i.e. univalent mappings.

Solution. For each $x \in A$ there are four options for $F(x)$: $\mathtt{0}, \mathtt{1}, \mathtt{2}$ and *undefined*. So there are $4 \times 4 = 16$ partial-functions from A to B.

(a) Total-functions.

Solution. For each $x \in A$ we have three options for $F(x)$: $\mathtt{0}, \mathtt{1}$ and 2. So there are $3 \times 3 = 9$ total-functions from A to B.

(b) Total mappings. [Hint: Consider the second solution to (i), but now $\emptyset$ is not an an acceptable output-set.]

Solution. For each $x \in A$ there are 7 options for the output-set, given that $\emptyset$ is excluded. So altogether we have $7^3 = 343$ total mappings.

(c) Surjective mappings. [Hint: Use (b)]

Solution. The surjective mappings from A to B are a mirror image of the total mappings from B to A. From (b) the number of such total mappings, when both domain and 5range have 3 elements, is 343.

(d) Injective mappings. [Hint: Same as the number of univalent mappings from B to A. Now (ii).]

Solution. The injective mappings from A to B are a mirror image of the partial functions from B to A. From (ii) the number of those, when both domain and range have 3 elements, is 12.

(e) Bijections. [Hint: This is a trick question.]

Solution. None, since A and B have different size.

2. (10+10+5%). Let $f : \mathbb{N} \to A$ be an injection and B an arbitrary set.

(a) Define an injection $g : \mathbb{N} \times B \to A \times B$.

Solution. For $x \in \mathbb{N}$ and $y \in B$ let $g(\langle x, y\rangle) = \langle f(x), y\rangle$. g is injective, because if $g(\langle x', y'\rangle) = g(\langle x, y\rangle)$ i.e. $\langle f(x'), y'\rangle = \langle f(x), y\rangle$ then $x' = x$ since f is injective and $y' = y$ by the definition of ordered pairs.

(b) Define an injection $h : \mathcal{P}(\mathbb{N}) \to \mathcal{P}(A)$.

Solution. For $A \subseteq \mathbb{N}$ let $h(X) = \{f(n) \mid n \in X\}$. h is a total-function, since it is uniquely defined for every input $X \subset \mathbb{N}$. h is injective because if $X \neq Y$, say $k \in X - Y$, then by the definition of h $f(k) \in h(X)$ but $f(k) \notin h(Y)$, so $h(X) \neq h(Y)$.

(c) Define a surjective partial-function $j : A \rightharpoonup \mathbb{N}$.

Solution. Let j be the inverse f^{-1} of f. In other words, $j(x)$ is defined to be the unique y for which $f(y) = x$, if there is such an y, and undefined otherwise.

Since f is injective, j is univalent, and so a partial-function. It is surjective on $\mathbb{N}$ because f is total.

3. (10+5%) Functions f, g over $\mathbb{N}$ are *almost equal* (notation: $f =_{ae} g$) if there are only finitely many n's for which $f(n) \neq g(n)$.

(a) Prove that $=_{ae}$ is an equivalence relation.

Solution. $=_{ae}$ is reflexive, since a function differs from itself on the empty set, which is finite.
It is symmetric by its very definition.
To see that it is transitive, suppose that $f =_{ae} g$, with f and g differing only over a finite set $A \subset \mathbb{N}$; and $g =_{ae} h$, say g and h differing only over a finite set B. Then f and h differ at most over $A \cup B$, which is finite. So $f =_{ae} h$.

(b) What is the equivalence class of the constant function $f(x) = 0$?

Solution. The collection of functions $g : \mathbb{N} \to \mathbb{N}$ for which $\{x \in \mid g(x) \neq 0\}$ is finite.

4. (10%), Prove that $\mathbb{R} \times \mathbb{R} \preccurlyeq \mathbb{R}$. [Hint: Define an injection $j : (0..1) \times (0..1) \rightarrow (0..1)$ where $j(a, b)$ has the infinite binary expansion obtained by merging the binary expansions of a and of b.]

Solution. Define $j : (0..1) \times (0..1) \rightarrow (0..1)$ where $j(a, b)$ $(a, b \in (0..1))$, given infinite decimal expansions $0.d_1 d_2 \cdot$ and $0.e_1 e_2 \cdots$ for a and b, is the real number with decimal expansion $0.d_1 e_1 d_2 e_2 \cdots$.

j is well-defined because

- Every $x \in (0..1)$ has a unique infinite expansion, for example $1/5$ expands to $0.19999\cdots$ and $1/6$ to $0.166666\cdots$.
- The merging operation is well-defined expansion yielding a unique real number.

We therefore have:

$\mathbb{R} \times \mathbb{R}$	$\preccurlyeq$	$\mathbb{R} \times (0..1)$	as in 2(a), since $\mathbb{R} \cong (0..1))$		
	$\preccurlyeq$	$(0..1) \times (0..1)$	for the same reason		
	$\preccurlyeq$	$(0..1)$	by j	$\preccurlyeq$	$\mathbb{R}$

Conversely, $\mathbb{R} \preccurlyeq \mathbb{R} \times \mathbb{R}$ by the injection $x \mapsto \langle 0, x \rangle$.

By the CBS Theorem it follows that $\mathbb{R} \times \mathbb{R} \cong \mathbb{R}$.

5. (10%) Show that the set F of functions from $\mathbb{N}$ to $\mathbb{N}$ is not countable.

Solution. Let $j : (0..1) \rightarrow F$ be the function that maps $a \in (0..1)$, with infinite decimal expansion $0.d_1 d_2 \cdots$, to the function $g_a : \mathbb{N} \rightarrow \{0, 1, 2...9\}$ defined by $g_a(n) = d_n$. j is an injection, because every $a \in (0..1)$ has a unique infinite decimal expansion.

So $(0..1) \preccurlyeq F$, and F is not countable, lest $(0..1)$ would be countable, which it is not.

6. (15%) Use the CBS Theorem to show that $\{\mathtt{a}, \mathtt{b}\}^* \cong \{\mathtt{a}, \mathtt{b}, \mathtt{c}\}^*$.
[Hint: An injection $h : \{\mathtt{a}, \mathtt{b}, \mathtt{c}\}^* \rightarrow \{\mathtt{a}, \mathtt{b}\}^*$ can be defined using a two-letter codes for $\mathtt{a}, \mathtt{b}$ and $\mathtt{c}$. (This is analogous to the binary coding of ASCII characters.)]

Solution. We have $\{\mathtt{a}, \mathtt{b}\}^* \preccurlyeq \{\mathtt{a}, \mathtt{b}, \mathtt{c}\}^*$ since the identity function on $\{\mathtt{a}, \mathtt{b}\}^*$ is an injection into $\{\mathtt{a}, \mathtt{b}, \mathtt{c}\}^*$.

Conversely, define $f : \{\mathtt{a}, \mathtt{b}, \mathtt{c}\}^* \rightarrow \{\mathtt{a}, \mathtt{b}\}^*$ by

$f(w) =_{\mathrm{df}} w$ with each $\mathtt{a}$ replaced by $\mathtt{aa}$, $\mathtt{b}$ by $\mathtt{bb}$, and $\mathtt{c}$ by $\mathtt{ab}$.

f is an injection:
For every string u the string $f(u)$ has length $2\,|u|$. So if $f(u) = f(v)$ then $|u| = |v|$, and if $u = \sigma_0 \cdot \cdots \cdot \sigma_k$ and $v = \tau_0 \cdot \cdots \cdot \tau_m$ then $k = m$, $f(u) = f(\sigma_0) \cdot \cdots \cdot f(\sigma_k)$, and $f(v) = f(\tau_0) \cdot \cdots \cdot f(\tau_k)$.

By the definition of f, $f(\mathtt{a}), f(\mathtt{b})$ and $f(\mathtt{c})$ are all different, so $\sigma_i = \tau_i$ for $i = 1..k$, in other words $u = v$. Thus $\{\mathtt{a}, \mathtt{b}, \mathtt{c}\}^* \preccurlyeq \{\mathtt{a}, \mathtt{b}\}^*$.
$\{\mathtt{a}, \mathtt{b}\}^* \cong \{\mathtt{a}, \mathtt{b}, \mathtt{c}\}^*$ follows by the CBS Theorem.