Assignment 1: Languages and sets

(Aug 28, 2024. Due on Canvas by 11:59pm, F Sep 6)

1. (8%) Let $L = \{\mathtt{0}, \mathtt{01}\}$ and $M = \{\mathtt{1}, \mathtt{12}\}$. So, for example, $L \cdot M = \{\mathtt{01}, \mathtt{012}, \mathtt{011}, \mathtt{0112}\}$. Exhibit similarly the languages $M \cdot L$, L^2, M^2, and L^3.

2. (12%) Let $\Sigma = \{\mathtt{a}, \mathtt{b}\}$. The language $A = (\Sigma^2)^*$ can be described in words as *"the set of Σ-strings of even length".* Describe in words, in a similar way, the following languages. Try to give, as much as possible, concise and clear descriptions.

   (a) $B = \{\mathtt{a}\} \cdot \Sigma^*$

   (b) $C = \Sigma^* \cdot \{\mathtt{b}\}$

   (c) $D = B \cap C$, where B and C are as above.

   (d) $E = \Sigma \cdot (\Sigma^2)^*$.

3. (15%) The previous problem asks for verbal descriptions of languages given by set and language operations. Here we do the opposite. Each of the following is an informal description of a language $L \subseteq \Sigma^*$ where $\Sigma = \{\mathtt{a}, \mathtt{b}\}$. Show how L can be defined using set operations (union, intersection, difference) and language operations (concatenation, star). For example, the set of strings that start with an a is $\{\mathtt{a}\} \cdot \Sigma^*$.

   (a) The set of strings that start with a and end with b.

   (b) The set of strings with a b.

   (c) The set of strings with exactly one b.

   (d) The set of strings with exactly 2 b's.

   (e) The set of strings of length $\geqslant 2$ that start and end with the same letter.

4. (6+8%) Suppose that a language L is closed under reversal, that is for every $w \in L$ we also have $w^R \in L$.

   (a) Show that $L \cdot L$ is also closed under reversal. [Hint: We know that $(x \cdot y)^R = y^R \cdot x^R$]

   (b) Show that L^* is closed under reversal.

5. (15%) Let $\Sigma = \{\mathtt{a}, \mathtt{b}\}$. Show that if $\mathtt{a} \cdot w = w \cdot \mathtt{a}$ then w is a string of a's.

6. (16%) Consider subsets of a fixed set U. Recall that the complement (in U) of a set $A \subseteq U$ is defined as $U - A$, and denoted by $\bar{A}$. We can define then intersection of subsets of U in terms of union and complement:
   $A \cap B = \overline{\bar{A} \cup \bar{B}}$

   (a) Define set-difference in terms of intersection and complement.

   (b) Define intersection in terms of just difference.

7. (20%) For each of the following languages L over the alphabet $\{\mathtt{a}, \mathtt{b}\}^*$ list its residues. For example, if $L = \{\mathtt{ab}, \mathtt{ba}, \mathtt{aaba}\}$ then the residues are

$$\begin{array}{rcll} L/\varepsilon &=& L & \\ L/\mathtt{a} &=& \{\mathtt{b}, \mathtt{aba}\} & \\ L/\mathtt{b} = L/\mathtt{aab} &=& \{\mathtt{a}\} & \\ L/\mathtt{ab} = L/\mathtt{ba} = L/\mathtt{aaba} &=& \{\varepsilon\} & \\ L/\mathtt{aa} &=& \{\mathtt{ba}\} & \\ L/w &=& \emptyset & \text{for any other } w \end{array}$$

And for $L = \mathcal{L}(\mathtt{aba}^*)$:

$$\begin{array}{rcll} L/\varepsilon &=& L & \\ L/\mathtt{a} &=& \mathcal{L}(\mathtt{ba}^*) & \\ L/\mathtt{ab}u &=& \mathcal{L}(\mathtt{a}^*) & \text{for any } u \in \mathcal{L}(\mathtt{a}^*) \\ L/w &=& \emptyset & \text{for any other } w \end{array}$$

(a) $L = \{\mathtt{a}, \mathtt{b}\}^*$ as a language over the alphabet $\{\mathtt{a}, \mathtt{b}, \mathtt{c}\}$.

(b) $L = \{\mathtt{a}^n\mathtt{b}\mathtt{a}^n \mid n \geqslant 0\}$.