Assignment 1: Languages and sets

(Aug 28, 2024. Due on Canvas by 11:59pm, F Sep 6)

1. (8%) Let $L = \{\mathtt{0}, \mathtt{01}\}$ and $M = \{\mathtt{1}, \mathtt{12}\}$. So, for example, $L \cdot M = \{\mathtt{01}, \mathtt{012}, \mathtt{011}, \mathtt{0112}\}$. Exhibit similarly the languages $M \cdot L$, L^2, M^2, and L^3.

   Solution.

$$\begin{aligned} M \cdot L &= \{\mathtt{10}, \mathtt{101}, \mathtt{120}, \mathtt{1201}\} \\ L^2 &= \{\mathtt{00}, \mathtt{001}, \mathtt{010}, \mathtt{0101}\} \\ M^2 &= \{\mathtt{11}, \mathtt{112}, , \mathtt{121}, \mathtt{1212}\} \\ L^3 &= \{\mathtt{000}, \mathtt{0001}, \mathtt{0010}, \mathtt{00101}, \mathtt{0100}, \mathtt{01001}, \mathtt{01010}, \mathtt{010101}\} \end{aligned}$$

2. (12%) Let $\Sigma = \{\mathtt{a}, \mathtt{b}\}$. The language $A = (\Sigma^2)^*$ can be described in words as *"the set of* Σ*-strings of even length".* Describe in words, in a similar way, the following languages. Shoot for concise and clear descriptions.

   (a) $B = \{\mathtt{a}\} \cdot \Sigma^*$

   Solution. The set of strings that start with an `a`.

   (b) $C = \Sigma^* \cdot \{\mathtt{b}\}$

   Solution. The set of strings that end with `b`.

   (c) $D = B \cap C$, where B and C are as above.

   Solution. The set of strings that start with `a` and end with `b`.

   (d) $E = \Sigma \cdot (\Sigma^2)^*$.

   Solution. The set of strings of odd length.

3. (15%) The previous problem asks for verbal descriptions of languages given by set and language operations. Here we do the opposite. Each of the following is an informal description of a language $L \subseteq \Sigma^*$ where $\Sigma = \{\mathtt{a}, \mathtt{b}\}$. Show how L can be defined using set operations (union, intersection, difference) and language operations (concatenation, star). For example, the set of strings that start with an `a` is $\{\mathtt{a}\} \cdot \Sigma^*$.

   (a) The set of strings that start with `a` and end with `b`.

   Solution. $\{\mathtt{a}\} \cdot \Sigma^* \cdot \{\mathtt{b}\}$

   (b) The set of strings with a `b`.

   Solution. $\Sigma^* \cdot \{\mathtt{b}\} \cdot \Sigma^*$.

   (c) The set of strings with exactly one `b`.

   Solution. $\{\mathtt{a}\}^* \cdot \{\mathtt{b}\} \cdot \{\mathtt{a}\}^*$.

   (d) The set of strings with exactly two `b`'s.

   Solution. $L = \{\mathtt{a}\}^* \cdot (\{\mathtt{b}\} \cdot \{\mathtt{a}\}^*)^2 = \{\mathtt{a}\}^* \cdot \{\mathtt{b}\} \cdot \{\mathtt{a}\}^* \cdot \{\mathtt{b}\} \cdot \{\mathtt{a}\}^*$

   (e) The set of strings of length $\geqslant 2$ that start and end with the same letter.

   Solution. $(\{\mathtt{a}\} \cdot \Sigma^* \cdot \{\mathtt{a}\}) \cup (\{\mathtt{b}\} \cdot \Sigma^* \cdot \{\mathtt{b}\})$

4. (6+8%) Suppose that a language L is closed under reversal: $L^R = L$.

   (a) Show that $L \cdot L$ is also closed under reversal, i.e. $(L \cdot L)^R = L \cdot L$. [Hint: You may use without proof the identity $(x \cdot y)^R = y^R \cdot x^R$]
   Solution.

$$\begin{array}{rcll} (L \cdot L)^R & = & \{x \cdot y \mid x, y \in L\}^R & \\ & = & \{(x \cdot y)^R \mid x, y \in L\} & \text{dfn of language reversal} \\ & = & \{y^R \cdot x^R \mid x, y \in L\} & \text{given} \\ & = & \{u \cdot v \mid u, v \in L^R\} & \\ & = & \{u \cdot v \mid u, v \in L\} & \text{since } L = L^R \\ & = & L \cdot L & \end{array}$$

   (b) Show that L^* is closed under reversal.
   Solution. From the hint above it also follows that for all strings $x_1, \ldots, x_k$ we have $(x_1 \cdot \cdots \cdot x_k)^R = {x_k}^R \cdot \cdots \cdot {x_1}^R$. So

$$\begin{array}{rcll} (L^*)^R & = & \{x_1 \cdot \cdots \cdot x_k \mid x_1, \ldots, x_k \in L\}^R & \\ & = & \{(x_1 \cdot \cdots \cdot x_k)^R \mid x_1, \ldots, x_k \in L\} & \text{dfn of language reversal} \\ & = & \{{x_k}^R \cdot \cdots \cdot {x_1}^R \mid x_1, \ldots, x_k \in L\} & \text{given} \\ & = & \{u_k \cdot \cdots \cdot u_1 \mid u_1, \ldots, u_k \in L^R\} & \\ & = & \{u_k \cdot \cdots \cdot u_1 \mid u_1, \ldots, u_k \in L\} & \text{since } L^R = L \\ & = & L^* & \text{given closure under reversal} \end{array}$$

5. (15%) Let $w \in \{\mathtt{a}, \mathtt{b}\}^*$. Prove that if $\mathtt{a} \cdot w = w \cdot \mathtt{a}$ then w is a string of $\mathtt{a}$'s.

   Solution. Suppose w is not a string of $\mathtt{a}$'s. We derive a contradiction as follows. ε is a string of $\mathtt{a}$'s. If $w \neq \varepsilon$ then it starts with an $\mathtt{a}$ and contains a $\mathtt{b}$. That is, w is $\mathtt{a}^n \cdot \mathtt{b} \cdot u$ for some $n > 0$ and $u \in \Sigma^*$. Using repeatedly $\mathtt{a} \cdot w = w \cdot \mathtt{a}$ we get $w = \mathtt{b} \cdot u \cdot \mathtt{a}^n$. So w starts with $\mathtt{b}$, a contradiction.

6. (16%) Consider subsets of a fixed set U. Recall that the complement (in U) of a set $A \subseteq U$ is defined as $U - A$, and denoted by $\bar{A}$. We can define then intersection of subsets of U in terms of union and complement: $A \cap B = \overline{\bar{A} \cup \bar{B}}$.

   (a) Define set-difference in terms of intersection and complement.
   Solution. $A - B = A \cap \bar{B}$.

   (b) Define intersection in terms of difference, and nothing else.
   Solution. $A \cap B = A - (A - B)$.

   For example, for $A \cup \emptyset = A$ we have

$$\begin{array}{rlll} x \in A \cup \emptyset & \text{IFF} & x \in A \text{ or } x \in \emptyset & \\ & \text{IFF} & x \in A & \text{since } x \in \emptyset \text{ is impossible} \end{array}$$

   Similarly $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ is proved by spelling out in our discourse the meanning of the operators:

$$\begin{array}{rll} x \in A \cap (B \cup C) & \text{IFF} & x \in A \text{ and either } x \in B \text{ or } x \in C \\ & \text{IFF} & \text{either } x \in A \text{ and } x \in B \text{ or } x \in A \text{ and } x \in C \\ & \text{IFF} & \text{either } x \in A \cap B \text{ or } x \in A \cap C \\ & \text{IFF} & x \in (A \cap B) \cup (A \cap C) \end{array}$$

(a) $A \cap \emptyset = \emptyset$
Solution.

$$\begin{array}{lll} x \in A \cap \emptyset & \text{IFF} & x \in A \text{ and } x \in \emptyset \\ & & \text{which cannot happen, by defn of } \emptyset \\ & \text{IFF} & x \in \emptyset \end{array}$$

(b) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
Solution.

$$\begin{array}{lll} x \in A \cup (B \cap C) & \text{i.e.} & x \text{ in } A \text{ or in both } B \text{ and } C \\ & \text{IFF} & x \text{ in both } A \text{ and } B \text{ or in both } A \text{ and } C \\ & \text{IFF} & x \in (A \cup B) \cap (A \cup C) \end{array}$$

7. (20%) For each of the following languages L over the alphabet $\{\mathtt{a}, \mathtt{b}\}^*$ list its residues. For example, if $L = \{\mathtt{ab}, \mathtt{ba}, \mathtt{aaba}\}$ then the residues are

$$\begin{array}{rcll} L/\varepsilon & = & L & \\ L/\mathtt{a} & = & \{\mathtt{b}, \mathtt{aba}\} & \\ L/\mathtt{b} = L/\mathtt{aab} & = & \{\mathtt{a}\} & \\ L/\mathtt{ab} = L/\mathtt{ba} = L/\mathtt{aaba} & = & \{\varepsilon\} & \\ L/\mathtt{aa} & = & \{\mathtt{ba}\} & \\ L/w & = & \emptyset & \text{for any other } w \end{array}$$

And $L = \mathcal{L}(\mathtt{aba}^*)$ has four residues.

$$\begin{array}{rcll} L/\varepsilon & = & L & \\ L/\mathtt{a} & = & \mathcal{L}(\mathtt{ba}^*) & \\ L/\mathtt{ab}u & = & \mathcal{L}(\mathtt{a}^*) & \text{for any } u \in \mathcal{L}(\mathtt{a}^*) \\ L/w & = & \emptyset & \text{for any other } w \end{array}$$

(a) $L = \{\mathtt{a}, \mathtt{b}\}^*$ as a language over the alphabet $\{\mathtt{a}, \mathtt{b}, \mathtt{c}\}$.
Solution. $L/w = L$ for all strings $w \in \{\mathtt{a}, \mathtt{b}\}^*$. $L/w = \emptyset$ if $\mathtt{c}$ occurs in w.

(b) $L = \{\mathtt{a}^n\mathtt{b}\mathtt{a}^n \mid n \geqslant 0\}$.
Solution. We have infinitely many residues:

$$\begin{array}{rcll} L/\mathtt{a}^i & = & \{\mathtt{a}^k\mathtt{b}\mathtt{a}^{i+k} \mid k \geqslant 0\} & \text{for every } i \geqslant 0. \\ L/\mathtt{a}^i\mathtt{b}\mathtt{a}^j & = & \{\mathtt{a}^{i-j}\} & \text{for every } i \geqslant j \geqslant 0 \\ L/w & = & \emptyset & \text{for all other strings } w \end{array}$$